OPERATORS ON POLYNOMIALS

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Contents

CHAPTER 1

Symmetric Functions

1.1. Elementary symmetric functions

An alphabet $\mathbb{A} = \{a_1, a_2, \ldots\}$ is a totally ordered set of indeterminates. One writes $\mathbb{A} + \mathbb{B}$ for the disjoint union of two alphabets.

Taking an extra indeterminate z, one has three fundamental series
(1.1.1)

$$\lambda_z(\mathbb{A}) := \prod_{a\in\mathbb{A}}(1+za) \ , \ \sigma_z(\mathbb{A}) := \prod_{a\in\mathbb{A}} \frac{1}{1-za} \ , \ \Psi_z(\mathbb{A}) := \sum_{i=1}^{\infty}\sum_{a\in\mathbb{A}} z^i a^i/i \, ,$$

the expansion of which gives the *elementary symmetric functions* $\Lambda^i(\mathbb{A})$, the *complete functions* $S^i(\mathbb{A})$, and the *power sums* $\Psi_i(\mathbb{A})$:
(1.1.2)

$$\lambda_z(\mathbb{A}) = \sum z^i\, \Lambda^i(\mathbb{A}) \ , \ \sigma_z(\mathbb{A}) = \sum z^i\, S^i(\mathbb{A}) \ , \ \boldsymbol{\Psi}_z(\mathbb{A}) = \sum_{i=1}^{\infty} z^i \Psi_i(\mathbb{A})/i \ .$$

Since $\log(1/(1-a)) = \sum_{i>0} a^i/i$, one has

$$(1.1.3) \qquad \sigma_z(\mathbb{A}) = \exp\left(\boldsymbol{\Psi}_z(\mathbb{A})\right) \quad , \quad \boldsymbol{\Psi}_z(\mathbb{A}) = \log\left(\sigma_z(\mathbb{A})\right) \ .$$

Addition of alphabets implies product of generating series

$$(1.1.4) \qquad \lambda_z(\mathbb{A}+\mathbb{B}) = \lambda_z(\mathbb{A})\, \lambda_z(\mathbb{B}) \quad , \quad \sigma_z(\mathbb{A}+\mathbb{B}) = \sigma_z(\mathbb{A})\, \sigma_z(\mathbb{B}) \ .$$

At the level of the individual symmetric functions, (1.1.4) gives

$$(1.1.5) \qquad \Lambda^n(\mathbb{A}+\mathbb{B}) = \sum_{i+j=n} \Lambda^i(\mathbb{A})\Lambda^j(\mathbb{B}) \ ,$$

$$(1.1.6) \qquad S^n(\mathbb{A}+\mathbb{B}) = \sum_{i+j=n} S^i(\mathbb{A})S^j(\mathbb{B}) \ .$$

Since one can invert formal series beginning by 1, or take any power of them, one can extend (1.1.1) by setting :

$$(1.1.7) \quad \sigma_z(\mathbb{A}-\mathbb{B}) := \frac{\prod_{b\in\mathbb{B}}(1-zb)}{\prod_{a\in\mathbb{A}}(1-za)} \quad , \quad \sigma_z(c\,\mathbb{A}) = \left(\sigma_z(\mathbb{A})\right)^c \ , \ c\in\mathbb{C} \ .$$

When $\mathbb{B} = 0$, or $\mathbb{A} = 0$, one recovers the two series $\sigma_z(\mathbb{A})$ and $\sigma_z(-\mathbb{B}) = \lambda_{-z}(\mathbb{B})$.

Notice that the addition of alphabets satisfies the usual properties of addition : $\sigma_z(-\mathbb{A})$ is the inverse of $\sigma_z(\mathbb{A})$ because $\mathbb{A} - \mathbb{A} = 0$ and $\sigma_z(0) = 1$. Similarly, the identity $(\mathbb{A}+\mathbb{C})-(\mathbb{B}+\mathbb{C}) = \mathbb{A}-\mathbb{B}$ translates, at the level of generating series, the fact that

$$\frac{\prod_b(1-zb)\prod_c(1-zc)}{\prod_a(1-za)\prod_c(1-zc)} = \frac{\prod_b(1-zb)}{\prod_a(1-za)} \,,$$

and nobody will deny that one may simplify a factor common to the numerator and denominator of a rational function !

Writing $\mathbb{A} + \mathbb{B}$ for the disjoint union of alphabets forces us to consider a finite alphabet as the sum of its sub-alphabets of cardinality 1, i.e to identify $\mathbb{A}$ and $\sum_{a\in\mathbb{A}} a$, and write

$$S^k(a_1 + a_2 + \cdots + a_n - b_1 - \cdots - b_m)$$

instead of $S^k(\mathbb{A} - \mathbb{B})$, when we shall need the letters composing the finite alphabets $\mathbb{A}$ and $\mathbb{B}$.

In particular, an indeterminate x has to be considered as an alphabet of cardinality 1. Very often, we shall use symmetric functions in $\mathbb{A} + x$ or $\mathbb{A} - x$.

1.2. Matrix Generating Functions

Let z stand now for the infinite matrix with diagonal $j - i = 1$ filled with 1's, all other entries being 0's.

Since z^k, $k \in \mathbb{N}$, is the matrix with 1's in the k-th diagonal above the main diagonal, and 0 outside of it, we see that now $\sigma_z(\mathbb{A})$ is a Toeplitz matrix (i.e. a matrix with constant values in each diagonal), that we shall denote by $\mathbb{S}(\mathbb{A})$; similarly $\lambda_z(\mathbb{A})$ is a matrix denoted $\mathbb{L}(\mathbb{A})$:

$$\mathbb{S}(\mathbb{A}) = \Big[S^{j-i}(\mathbb{A})\Big]_{i,j\geq 0} \quad \& \quad \mathbb{L}(\mathbb{A}) = \Big[\Lambda^{j-i}(\mathbb{A})\Big]_{i,j\geq 0} \,. \tag{1.2.1}$$

$$\mathbb{S}(\mathbb{A}) = \begin{bmatrix} S^0(\mathbb{A}) & S^1(\mathbb{A}) & S^2(\mathbb{A}) & S^3(\mathbb{A}) & \cdots \\ S^{-1}(\mathbb{A}) & S^0(\mathbb{A}) & S^1(\mathbb{A}) & S^2(\mathbb{A}) & \cdots \\ S^{-2}(\mathbb{A}) & S^{-1}(\mathbb{A}) & S^0(\mathbb{A}) & S^1(\mathbb{A}) & \cdots \\ & \ddots & \ddots & \ddots & \end{bmatrix}$$

$$\mathbb{L}(\mathbb{A}) = \begin{bmatrix} \Lambda^0(\mathbb{A}) & \Lambda^1(\mathbb{A}) & \Lambda^2(\mathbb{A}) & \Lambda^3(\mathbb{A}) & \cdots \\ \Lambda^{-1}(\mathbb{A}) & \Lambda^0(\mathbb{A}) & \Lambda^1(\mathbb{A}) & \Lambda^2(\mathbb{A}) & \cdots \\ \Lambda^{-2}(\mathbb{A}) & \Lambda^{-1}(\mathbb{A}) & \Lambda^0(\mathbb{A}) & \Lambda^1(\mathbb{A}) & \cdots \\ & \ddots & \ddots & \ddots & \end{bmatrix} .$$

These matrices are upper triangular, but it is wiser to write entries S^{-k} rather than their value 0.

Addition or subtraction of alphabets still correspond to product of generating functions, whether z is an indeterminate or a matrix makes no difference :

$$\mathbb{S}(\mathbb{A} \pm \mathbb{B}) = \mathbb{S}(\mathbb{A})\,\mathbb{S}(\mathbb{B})^{\pm 1} \quad \& \quad \mathbb{L}(\mathbb{A} \pm \mathbb{B}) = \mathbb{L}(\mathbb{A})\,\mathbb{L}(\mathbb{B})^{\pm 1} \ . \tag{1.2.2}$$

Indeed, formulas (1.1.5) and (1.1.6) exactly tells that an entry of the matrix $\mathbb{S}(\mathbb{A} \pm \mathbb{B})$ (resp. $\mathbb{L}(\mathbb{A} \pm \mathbb{B})$) is a sum of products of entries of $\mathbb{S}(\mathbb{A}$ and $\mathbb{S}(\pm\mathbb{B})$ (resp. $\mathbb{L}(\mathbb{A}$ and $\mathbb{L}(\pm\mathbb{B})$) of appropriate indices.

1.3. Schur Functions

The advantage of matrices, compared to formal series, is that they have minors, that we shall index by (increasing) partitions, or more generally, by vectors with components in $\mathbb{Z}$. More precisely, given $I = (i_1, \ldots, i_n) \in \mathbb{Z}^n$, $J = (j_1, \ldots, j_n) \in \mathbb{Z}^n$ one defines the *skew Schur function* $S_{J/I}(\mathbb{A})$ to be the minor of $\mathbb{S}(\mathbb{A})$ taken on rows $i_1 + 1, i_2 + 2, \ldots, i_n + n$ and columns $j_1 + 1, \ldots, j_n + n$ (we define the minor to be 0 if one of these numbers is < 0). When $I = 0^n$, the minor is called a *Schur function* and one writes $S_J(\mathbb{A})$ instead of $S_{J/0^n}(\mathbb{A})$.

In other words,

$$S_{J/I}(\mathbb{A}) = \left| S^{j_k - i_h + k - h}(\mathbb{A}) \right|_{1 \le h,k \le n} \ . \tag{1.3.1}$$

The expression of a Schur function as a determinant of complete functions is called the *Jacobi-Trudi* determinant (we shall see that there is also another expression in terms of elementary symmetric functions).

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One can enter a (decreasing) partition or a skew partition.
Matrices are transposed along the antidiagonal,
compared to the conventions in the course.
ACE> SfJtMat([5,4,1]), SfJtMat([ [5,4,1],[2,1] ]);
                      [h5    h6    h7]   [h3    h5    h7]
                      [h3    h4    h5],  [h1    h3    h5]
                      [0     1     h1]   [0     0     h1]
```

One can visualize the Schur function $S_{J/I}$ as being obtained from the initial minor of the same order, by shifting the columns by J, and the rows by I :

$$\begin{array}{ccc|c} 0 & 1 & 2 & -i_1 \\ -1 & 0 & 1 & -i_2 \\ -2 & -1 & 0 & -i_3 \\ \hline j_1 & j_2 & j_3 & \end{array} \Rightarrow \begin{vmatrix} 0 + j_1 - i_1 & 1 + j_2 - i_1 & 2 + j_3 - i_1 \\ -1 + j_1 - i_2 & 0 + j_2 - i_2 & 1 + j_3 - i_2 \\ -2 + j_1 - i_3 & -1 + j_2 - i_3 & 0 + j_3 - i_3 \end{vmatrix} .$$

It is convenient to also use determinants in elementary symmetric functions :

$$\Lambda_{J/I}(\mathbb{A}) = \left| \Lambda^{j_k - i_h + k - h}(\mathbb{A}) \right|_{1 \le h,k \le n} . \tag{1.3.2}$$

Of course, one must not forget that $\Lambda^i(\mathbb{A}) = (-1)^i S^i(-\mathbb{A})$, $i \in \mathbb{Z}$, and thus the $\Lambda_{J/I}(\mathbb{A})$ are also skew Schur functions in $-\mathbb{A}$ (we shall see that they also are Schur functions in $\mathbb{A}$, but indexed by "column lengths").

To write easily a Schur function $S_J(\mathbb{A})$, one first writes the diagonal, then complete the columns, increasing or decreasing indices by 1 when moving up or down :

$$J = [1,2,4] \Rightarrow \begin{vmatrix} S_1(\mathbb{A}) & & \\ & S_2(\mathbb{A}) & \\ & & S_4(\mathbb{A}) \end{vmatrix} \Rightarrow \begin{vmatrix} S_1(\mathbb{A}) & S_3(\mathbb{A}) & S_6(\mathbb{A}) \\ S_0(\mathbb{A}) & S_2(\mathbb{A}) & S_5(\mathbb{A}) \\ S_{-1}(\mathbb{A}) & S_1(\mathbb{A}) & S_4(\mathbb{A}) \end{vmatrix} = S_{124}(\mathbb{A})$$

One needs to enlarge the definition of a Schur function to be able to play with different alphabets at the same time.

Given n, two sets of alphabets $\{\mathbb{A}_1, \mathbb{A}_2, \dots, \mathbb{A}_n\}$, $\{\mathbb{B}_1, \mathbb{B}_2, \dots, \mathbb{B}_n\}$, and $J \in \mathbb{N}^n$, we define the *multi-Schur function*

$$S_J(\mathbb{A}_1 - \mathbb{B}_1, \dots, \mathbb{A}_n - \mathbb{B}_n) := \left| S_{j_k + k - h}(\mathbb{A}_k - \mathbb{B}_k) \right|_{1 \le h,k \le n} . \tag{1.3.3}$$

In the case where the alphabets are repeated, we indicate by a semicolon the corresponding block separation : given $H \in \mathbb{Z}^p$, $K \in \mathbb{Z}^q$, then $S_{H;K}(\mathbb{A}-\mathbb{B}; \mathbb{C}-\mathbb{D})$ stands for the multi-Schur function with index the concatenation of H and K, and alphabets $\mathbb{A}_1 = \cdots = \mathbb{A}_p = \mathbb{A}$, $\mathbb{B}_1 = \cdots = \mathbb{B}_p = \mathbb{B}$, $\mathbb{A}_{p+1} = \cdots = \mathbb{A}_{p+q} = \mathbb{C}$, $\mathbb{B}_{p+1} = \cdots = \mathbb{B}_{p+q} = \mathbb{D}$.

To write a multi-Schur function easily, one first fills the diagonal, then completes columns by keeping the same alphabet in each column :

$$S_{12;4}(\mathbb{A};\, \mathbb{B}) \Rightarrow \begin{vmatrix} S_1(\mathbb{A}) & & \\ & S_2(\mathbb{A}) & \\ & & S_4(\mathbb{B}) \end{vmatrix} \Rightarrow \begin{vmatrix} S_1(\mathbb{A}) & S_3(\mathbb{A}) & S_6(\mathbb{B}) \\ S_0(\mathbb{A}) & S_2(\mathbb{A}) & S_5(\mathbb{B}) \\ S_{-1}(\mathbb{A}) & S_1(\mathbb{A}) & S_4(\mathbb{B}) \end{vmatrix} .$$

```
Enter an increasing partition, and a list of alphabets
SchurDrap:=proc(pa,lA)   local n,i,j,ma; n:=nops(pa);
 transpose(array([seq(
 map(proc(x,i,ll) if (x>0) then s[x](op(i,ll)) elif (x=0)
  then 1 else 0 fi end,[seq(pa[i]-j+i,j=1..n)],i,lA),i=1..n)] ));
 end:
ACE> SchurDrap([1,3,5], [A1,A1, A2-A3]);
                    [s[1](A1)    s[4](A1)    s[7](A2 - A3)]
                    [   1        s[3](A1)    s[6](A2 - A3)]
                    [   0        s[2](A1)    s[5](A2 - A3)]
```

Far from being a complication compared to usual symmetric functions in an alphabet only, multi-Schur functions allow easier induction, because they can be easily transformed.

Notice that for any $k \in \mathbb{N}$, any $\mathbb{A}$, any x, one has

$$S^k(\mathbb{A} - x) = S^k(\mathbb{A}) - x\, S^{k-1}(\mathbb{A}) \ .$$

Now, if in the determinant expressing a multi-Schur function of order n, one replaces in row i, $i < n$, (denote it r_i) all the alphabets $\mathbb{A}_k - \mathbb{B}_k$ by $\mathbb{A}_k - \mathbb{B}_k - x$, then r_i is transformed into $r_i - x r_{i+1}$, and the value of the determinant is unchanged.

Iterating the reasoning, given two indeterminates x, y, one can change in row i, $i < n-1$, all the alphabets $\mathbb{A}_k - \mathbb{B}_k$ by $\mathbb{A}_k - \mathbb{B}_k - x - y$, and the value of the determinant is constant because the transformation has changed r_i into $r_i - (x{+}y)\, r_{i+1} + xy\, r_{i+2}$. It is not difficult to give the general law, transforming rows by linear combination of rows below it[1]. In final, the following "transformation lemma" tells which subtraction are possible in a multi-Schur function :

LEMMA 1.3.1. *Let $S_J(\mathbb{A}_1 - \mathbb{B}_1, \ldots, \mathbb{A}_n - \mathbb{B}_n)$ be a multi-Schur function, and $\mathbb{D}_1, \mathbb{D}_2, \ldots, \mathbb{D}_{n-1}$ be a family of finite alphabets such that $card(\mathbb{D}_i) \leq n - i$, $1 \leq i \leq n-1$. Then $S_J(\mathbb{A}_1 - \mathbb{B}_1, \ldots, \mathbb{A}_n - \mathbb{B}_n)$ is equal to the determinant*

$$\left| \, S_{j_k+k-i}(\mathbb{A}_k - \mathbb{B}_k - \mathbb{D}_i) \, \right|_{1 \leq i,k \leq n}$$

Proof. Indeed the transformation has consisted in changing row i into the linear combination

$$r_i + r_{i+1}\, S^1(-\mathbb{D}_i) + r_{i+2}\, S^2(-\mathbb{D}_i) + \cdots + r_n\, S^{n-i}(-\mathbb{D}_i)$$

because, for any $\mathbb{A}, \mathbb{B}$, any $j \in \mathbb{Z}$, one has

$$S_j(\mathbb{A}-\mathbb{B}-\mathbb{D}_i) = S_j(\mathbb{A}-\mathbb{B}) + S_{j-1}(\mathbb{A}-\mathbb{B}) S^1(-\mathbb{D}_i) + \cdots + S_{j-n+i}(\mathbb{A}-\mathbb{B}) S^{n-i}(-\mathbb{D}_i) \ .$$

□

Thanks to the lemma, we can transform the expression of $S_J(\mathbb{A}-x)$.

LEMMA 1.3.2. *Let $J \in \mathbb{N}^n$, $k \in \mathbb{N}$, $\mathbb{A}, \mathbb{B}$ be arbitrary and x be an indeterminate. Then*

$$S_J(\mathbb{A}-\mathbb{B} - x)\, x^k = S_{j;k}(\mathbb{A}-\mathbb{B}; x) \ . \tag{1.3.4}$$

Proof. In the determinant on the right, subtract x to all rows except the bottom one. Then the last column becomes

$$S^{k+n}(0) = 0, \ \ldots, S^{k+1}(0) = 0, \ S^k(x) = x^k \ ,$$

[1] but remembering that the row index must not be greater than n. This is why one cannot subtract x in the last row, nor $x + y$ in row r_{n-1}.

and the cofactor of the entry x^k is indeed $S_J(\mathbb{A}-\mathbb{B}-x)$. □

1.4. Resultant

Schur functions indexed by *rectangular partitions*[2] play a special role. They appear in many classical topics of 19th century mathematics: elimination theory, continued fractions, etc. In particular, they allow to express the resultant of two polynomials in x, and we shall give many applications of this fact.

Given two finite alphabets $\mathbb{A} = \{a\}$, $\mathbb{B} = \{b\}$, of respective cardinalities α, β, let the *resultant* of $\mathbb{A}, \mathbb{B}$ be

$$R(\mathbb{A}, \mathbb{B}) := \prod_{a\in\mathbb{A}} \prod_{b\in\mathbb{B}} (a-b) \ .$$

PROPOSITION 1.4.1. *The resultant of two alphabets* $\mathbb{A}, \mathbb{B}$ *is equal to the Schur function*

$$S_{\beta^\alpha}(\mathbb{A}-\mathbb{B}) = (-1)^{\alpha\beta} S_{\alpha^\beta}(\mathbb{B}-\mathbb{A}) \ . \tag{1.4.1}$$

Proof. Choose a letter $a \in \mathbb{A}$ and subtract $(\mathbb{A}-a)$ from the top row of the determinant expressing $S_{\beta^\alpha}(\mathbb{A}-\mathbb{B})$. The determinant obtained is equal to the Schur function $S_{\beta^{\alpha-1};\beta}(\mathbb{A}-\mathbb{B}; a-\mathbb{B})$ (up to symmetry with respect to the middle, compared to the usual conventions). Now, one can subtract a from all the columns, except the first one. The first row has become

$$S_\beta(a-\mathbb{B}), S_{\beta+1}(-\mathbb{B}) = 0, \ldots, S_{\beta+\alpha-1}(-\mathbb{B}) = 0 \ ,$$

the zeroes being due to the fact[3] that $S^k(-\mathbb{B}) = 0$ for $k > \beta$. Therefore the new determinant factorizes into

$$S_\beta(a-\mathbb{B})\, S_{\beta^{\alpha-1}}((\mathbb{A}-a)-\mathbb{B}) = \prod_{b\in\mathbb{B}} (a-b)\, S_{\beta^{\alpha-1}}\big((\mathbb{A}-a)-\mathbb{B}\big) \ ,$$

and the result follows by induction on α.

[2]partitions with equal parts; we shall write m^n, or □ when m, n have been fixed.

[3]The elementary symmetric functions of a finite alphabet of degree higher than the cardinality are null.

For example, for $\alpha = 3$, $\beta = 5$, one has

$$S_{555}(\mathbb{A}-\mathbb{B}) = \begin{vmatrix} S^5(\mathbb{A}-\mathbb{B}) & S^6(\mathbb{A}-\mathbb{B}) & S^7(\mathbb{A}-\mathbb{B}) \\ S^4(\mathbb{A}-\mathbb{B}) & S^5(\mathbb{A}-\mathbb{B}) & S^6(\mathbb{A}-\mathbb{B}) \\ S^3(\mathbb{A}-\mathbb{B}) & S^4(\mathbb{A}-\mathbb{B}) & S^5(\mathbb{A}-\mathbb{B}) \end{vmatrix} =$$

$$\begin{vmatrix} S^5(a-\mathbb{B}) & S^6(a-\mathbb{B}) & S^7(a-\mathbb{B}) \\ S^4(\mathbb{A}-\mathbb{B}) & S^5(\mathbb{A}-\mathbb{B}) & S^6(\mathbb{A}-\mathbb{B}) \\ S^3(\mathbb{A}-\mathbb{B}) & S^4(\mathbb{A}-\mathbb{B}) & S^5(\mathbb{A}-\mathbb{B}) \end{vmatrix} = \begin{vmatrix} S^5(a-\mathbb{B}) & S^6(-\mathbb{B}) & S^7(-\mathbb{B}) \\ S^4(\mathbb{A}-\mathbb{B}) & S^5(\mathbb{A}-a-\mathbb{B}) & S^6(\mathbb{A}-a-\mathbb{B}) \\ S^3(\mathbb{A}-\mathbb{B}) & S^4(\mathbb{A}-a-\mathbb{B}) & S^5(\mathbb{A}-a-\mathbb{B}) \end{vmatrix} .$$

The simplest case of a resultant is when $\mathbb{A}$ has only one letter:

$$R(a, \mathbb{B}) = \prod_{b\in\mathbb{B}} (a-b) = S^\beta(a-\mathbb{B}) \ .$$

Note that

$$S^{\beta+i}(a-\mathbb{B}) = a^i \ , \quad S^\beta(a-\mathbb{B}) \ , \ i \geq 0 \ .$$

This is given by the generating function $\sigma_z(a-\mathbb{B})$, but more simply, one can add i letters to $\mathbb{B}$, and specialize them to 0.

1.5. Padé Approximants

Let us just give one application of rectangular Schur functions, the interpolation of functions in one variable $f(z)$ by rational functions $N(z)/D(z)$ of fixed degrees.

By "function", we mean, following Lagrange, a formal series in z which is supposed to be the Taylor expansion of a function in the neighbourhood of 0, and hopefully, even farther.

Forgetting the word "interpolation", and using alphabets, the problem we have to solve is the following :

given an alphabet $\mathbb{E}$, and two positive integers α, β, find two finite alphabets $\mathbb{A}$, $\mathbb{B}$: $|\mathbb{A}| = \alpha$, $|\mathbb{B}| = \beta$, such that

$$(1.5.1) \qquad \sigma_z(\mathbb{E}) \ \equiv \ \frac{N(z)}{D(z)} = \sigma_z(\mathbb{A}-\mathbb{B}) \qquad \mod z^{\alpha+\beta+1} \ .$$

In other words, we require that the two Taylor series coincide up to the highest possible degree, having $\alpha+\beta$ parameters at our disposal.

We could also say that we want to extract $\mathbb{A}$, then $\mathbb{B}$ from the first complete functions of $\mathbb{A}-\mathbb{B}$ (which are equal to those of $\mathbb{E}$).

This problem was solved by Cauchy, Jacobi, &c., but this is Padé who made a systematic study of it and took the patent.

Having promised an application of resultants, I must force them to appear. Instead of writing

$$\sigma_z(\mathbb{A}-\mathbb{B}) = \prod_{b\in\mathbb{B}}(1-zb) / \prod_{a\in\mathbb{A}}(1-za)$$

let us take $x = 1/z$ and multiply numerator and denominator by $R(\mathbb{A},\mathbb{B})$. Then

$$\frac{N(z)}{D(z)} = \pm x^{\alpha-\beta}\, \frac{R(x,\mathbb{B})R(\mathbb{A},\mathbb{B})}{R(x,\mathbb{A})R(\mathbb{B},\mathbb{A})} = \pm x^{\alpha-\beta}\, \frac{R(x+\mathbb{A},\mathbb{B}))}{R(x+\mathbb{B},\mathbb{A})}\ .$$

But now, our rational fraction has become the quotient of two resultants, and we can write it :

$$[\beta/\alpha] = \frac{N(z)}{D(z)} = \pm x^{\alpha-\beta}\, S_{\beta^{\alpha+1}}(\mathbb{A}+x-\mathbb{B})\, S_{(\beta+1)^\alpha}(\mathbb{A}-\mathbb{B}-x)^{-1}.$$

The two determinants involve only, apart from x, the complete functions $S^1(\mathbb{A}-\mathbb{B}),\ldots,S^{\alpha+\beta}(\mathbb{A}-\mathbb{B})$, that is, *exactly those which coincide with the coefficients of the series.*

Therefore, we have solved the problem, and obtained

$$\frac{N(z)}{D(z)} = \pm x^{\alpha-\beta}\, S_{\beta^{\alpha+1}}(\mathbb{E}+x)\, S_{(\beta+1)^\alpha}(\mathbb{E}-x)^{-1}\ , \tag{1.5.2}$$

Thanks to to the different expressions of the resultant, the Padé approximant can be written :

$$(1.5.3)\quad (-1)^{\alpha\beta+\alpha+\beta}\, \frac{z^\beta S_{(\alpha+1)^\beta}(\mathbb{B}-\mathbb{A}-\frac{1}{z})}{z^\alpha S_{(\beta+1)^\alpha}(\mathbb{A}-\mathbb{B}-\frac{1}{z})}$$
$$= (-1)^\alpha\, \frac{z^\beta S_{\beta^{\alpha+1}}(\mathbb{A}-\mathbb{B}+\frac{1}{z})}{z^\alpha S_{(\beta+1)^\alpha}(\mathbb{A}-\mathbb{B}-\frac{1}{z})} = (-1)^{\alpha\beta}\, \frac{z^\beta \Lambda_{(\alpha+1)^\beta}(\mathbb{A}-\mathbb{B}+\frac{1}{z})}{z^\alpha \Lambda_{(\beta+1)^\alpha}(\mathbb{B}-\mathbb{A}+\frac{1}{z})}$$

Do not forget that, for any $\mathbb{A}$, any x of rank 1, on has the decompositions $S^k(\mathbb{A}+x) = \sum_i x^i S^{k-i}(\mathbb{A})$, $S^k(\mathbb{A}-x) = S^k(\mathbb{A}) - xS^{k-1}(\mathbb{A})$, $\Lambda^k(\mathbb{A}+x) = \Lambda^k(\mathbb{A}) - x\Lambda^{k-1}(\mathbb{A})$. This shows how to expand the preceding functions in terms of x.

In fact, using our favorite transformations on Schur functions, we can restrict $1/z$ to only one column and write :

$$[\beta/\alpha] = (-1)^\alpha z^{\beta-\alpha}\, S_{\beta;\,\beta^\alpha}(\mathbb{A}-\mathbb{B}+\frac{1}{z};\mathbb{A}-\mathbb{B})\ /\ S_{(\beta+1)^\alpha\,;\,0}(\mathbb{A}-\mathbb{B};\frac{1}{z}) \tag{1.5.4}$$

These expressions can be found in the literature: (1.5.4) is due to Jacobi [**7**], while Sylvester [**30**, I n 57] gave the second expression of (1.5.3).

The explicit expansion of the numerator and the denominator of the Padé approximant, in terms of the Schur functions of $\mathbb{A}-\mathbb{B}$, is (1.5.5)

$$[\beta/\alpha] = \frac{S_{\beta^\alpha} + zS_{1\beta^\alpha} + z^2 S_{2\beta^\alpha} + \cdots + z^\beta S_{\beta\beta^\alpha}}{S_{\beta^\alpha} - zS_{\beta^{\alpha-1},\beta+1} + z^2 S_{\beta^{\alpha-2},(\beta+1)^2} + \cdots + (-z)^\alpha S_{(\beta+1)^\alpha}}\ .$$

For example, for $\beta = 1, \alpha = 1$, one gets

$$[1/1] = \frac{S_1 + zS_{11}}{S_1 - zS_2} = 1 + zS^1 + z^2S^2 + z^3S^{22}/S^1 + z^4S^{222}/S^{11} + \cdots ,$$

for $\beta = 1, \alpha = 2$,

$$[1/2] = \frac{S_{11} + zS_{111}}{S_{11} - zS_{12} + z^2S_{22}} = 1+zS^1+z^2S^2+z^3S^3+z^4\frac{S_{114}+S_{15}-S_{222}}{S_{11}}+\cdots ,$$

for $\beta = 2, \alpha = 1$,

$$[2/1] = \frac{S_2 + zS_{12} + z^2S_{22}}{S_2 - zS_3} = 1 + zS^1 + z^2S^2 + z^3S^3 + z^4\frac{S^{33}}{S^2} + \cdots$$

and for $\beta = 2, \alpha = 2$,

$$\frac{S_{22} + zS_{122} + z^2S_{222}}{S_{22} - zS_{23} + z^2S_{33}} = 1+zS^1+z^2S^2+z^3S^3+z^4S^4+\frac{S^{144} - 2S^{234} + S^{333}}{S_{22}}+\cdots$$

1.6. Cauchy Formulas

The resultant is a Schur function in a difference of alphabets, and consequently, a minor of a product of two matrices.

The Binet-Cauchy theorem for minors of the product of two matrices implies, in the case of the product

$$\mathbb{S}(\mathbb{A} - \mathbb{B}) = \mathbb{S}(\mathbb{A})\,\mathbb{S}(-\mathbb{B}) = \mathbb{S}(\mathbb{A})\,\mathbb{S}(\mathbb{B})^{-1}$$

the following expansion of skew-Schur functions:

$$(1.6.1)\quad S_{J/I}(\mathbb{A} + \mathbb{B}) = \sum\nolimits_K S_{J/K}(\mathbb{A})\, S_{K/I}(-\mathbb{B}) = \sum\nolimits_K (-1)^{|K/I|} S_{J/K}(\mathbb{A})\, S_{K^\sim/I^\sim}(\mathbb{B}) ,$$

sum over all partitions K.

Applying it to the resultant, one gets

$$R(\mathbb{A}, \mathbb{B}) := \prod_{a\in\mathbb{A},\, b\in\mathbb{B}} (a - b) = \sum_I S_I(\mathbb{A}) S_{\square/I}(-\mathbb{B})$$

$$(1.6.2)\qquad = \sum_I (-1)^{|\square/I|} S_I(\mathbb{A}) S_{\square^\sim/I^\sim}(\mathbb{B}) ,$$

where $\square = \beta^\alpha$, $\alpha = card(\mathbb{A})$, $\beta = card(\mathbb{B})$.

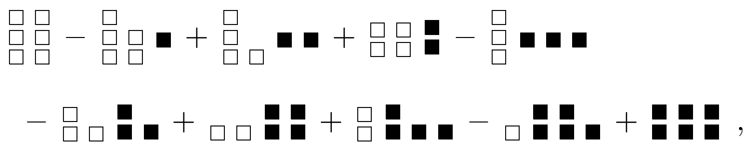
Taking inverse variables, and using the correspondence between the Schur functions in $a_1, a_2, \ldots$, and those of $1/a_1, 1/a_2, \ldots$, one can give the following second form of Cauchy formula :

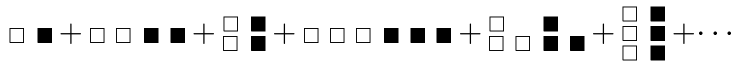
$$(1.6.3)\qquad \prod_{a\in\mathbb{A},\, b\in\mathbb{B}} (1 - ab) = \sigma_1(-\mathbb{A}\mathbb{B}) = \sum_I (-1)^{|I|} S_I(\mathbb{A}) S_{I^\sim}(\mathbb{B}) .$$

Now, the cardinalities have disappeared, the formula is valid for any infinite $\mathbb{A}, \mathbb{B}$, i.e. for two formal series, and one can replace $\mathbb{B}$ by $-\mathbb{B}$, thus obtaining the most common form of Cauchy formula :

$$\prod_{a\in\mathbb{A},\, b\in\mathbb{B}} (1-ab)^{-1} = \sigma_1(\mathbb{A}\mathbb{B}) = \sum_I S_I(\mathbb{A})S_I(\mathbb{B}) \ . \tag{1.6.4}$$

Coding Schur functions by diagrams of partitions, one can visualize (1.6.2) and (1.6.4) by pairs of complementary diagrams[4], or pairs of identical diagrams, white diagrams coding Schur functions in $\mathbb{A}$, black ones in $\mathbb{B}$:

$$\prod_{a,b}(a-b) =$$


$$\prod_{a,b}(1-ab)^{-1} = 1+$$


1.7. λ-rings

Schur functions are called Schur functions not because they were defined by Schur, but because it is considered that their true meaning appears in relation with representations of the linear group[5].

What we have already used in fact, is that Schur functions are *functors*, because we took as an argument formal expressions in alphabets like $\mathbb{A} - \mathbb{B}$, $(\mathbb{A}-\mathbb{B})(\mathbb{C}+\mathbb{D})$ instead of only sets of variables.

In fact, what we really want is to consider symmetric polynomials as *operators*. On what ? For the present, operating on polynomials will be enough. And since symmetric functions can be expressed as functions of power sums, it will be sufficient to define the *functor power-sum* Ψ_i, acting on polynomials in the following way :.

$$P = \sum_{\alpha,u} \alpha\, u \quad \Rightarrow \quad \Psi_i(P) = \sum_{\alpha,u} \alpha\, u^i \ , \tag{1.7.1}$$

[4]we have taken $\mathbb{A}, \mathbb{B}$ of cardinalities $3, 2$; the enumeration involves all pairs of partitions of complementary shape in the rectangle 3×2.

[5]But they also can be interpreted as Schubert cycles in the cohomology ring of Grassmann varieties, and it is a matter of taste to rate representation theory higher than geometry. In fact these two interpretations are closely related. As for concerns Schubert polynomials, the cohomological interpretation came also before representation considerations.

where $\alpha \in \mathbb{C}$, and u is monom.

It is a little surprising that such a simple rule can be of interest at all, but in fact it really implies the full theory of symmetric polynomials. The non-trivial point is that Ψ_i acts differently on constants and variables.

Using the generating functions of complete functions or elementary symmetric functions in terms of power sums, one can rewrite :

$$P = \sum_{\alpha,u} \alpha\, u \mapsto \lambda_z(P) = \prod (1 + zu)^{\alpha} \ , \tag{1.7.2}$$

$$P = \sum_{\alpha,u} \alpha\, u \mapsto \sigma_z(P) = \prod (1 - zu)^{-\alpha} \ . \tag{1.7.3}$$

Explicitly, the action of the elementary functors on constants α and variables x is :
(1.7.4)

$$\begin{cases} \Psi_i(\alpha) = \alpha \ , \quad S^i(\alpha) = \binom{\alpha+i-1}{i} \ , \qquad \Lambda^i(\alpha) = \binom{\alpha}{i} \\ \Psi_i(x) = x^i \ , \qquad S^i(x) = x^i \ , \qquad \Lambda^i(x) = 0,\ i > 1, \quad \Lambda^1(x) = x \end{cases}$$

Of course, a constant can be variable, and a variable can be specialized to a constant. Thus, in the present case, the terminology "constant" and "variable" or "indeterminate" is confusing. What we have in our ring, on which operate Ψ_i, S^i, Λ^i, are two basic objects: the *elements of binomial type* α, which are invariant under all the Ψ_i, and the *elements of rank* 1 which are annihilated by all Λ^i, $i > 1$ (and different from 0).

Notice that an element which is of binomial type and of rank 1 must be equal to 1. This has the consequence that one cannot specialize "variables" in a way compatible with the action of power sums, apart from specializing to 1, or 0.

Of course, one can use λ-rings to prove an algebraic identity $A = B$, and then use any specialization of both members A, B.

For example, given the rank-1 elements $x, y, \zeta, \alpha, \beta, \gamma$, the n-th Legendre polynomial $P_n(x)$ is the specialization of the following different elements :

$$\begin{aligned} P_n(x) &= S^n\big((y + y^{-1})/2\big) \\ &= S^n\big((n{+}1)x - n(\zeta{+}1)\big)\, 2^{-n} \\ &= S^n\big((n{+}1)b - n\zeta\big) \\ &= \Lambda^n\big(n - (n{+}1)c\big) \\ &= \Lambda^n(na + nb) \ , \end{aligned}$$

in

$$y = x + \sqrt{x^2-1}\ ,\ \zeta = -1\ ,\ a = \frac{x+1}{2}\ ,\ b = \frac{x-1}{2}\ ,\ c = \frac{1-x}{2}\ .$$

Let us just write the different expressions for $n = 2$, leaving as an exercise the general case.

$S^2((y+y^{-1})/2)$ expands, thanks to Cauchy formula, into

$$S_2(1/2)S_2(y+y^{-1}) + S_11(1/2)S_11(y+y^{-1})$$
$$= 3/8(y^2+1+1/y^2) - 1/8y/y\ ,$$

$$S^2(3x-2(\zeta+1)) = S^2(3x) - (3x)(2\zeta+2) + \Lambda^2(2\zeta+2)$$
$$= 1 - 6x + 4\zeta + \zeta^2 - 6x\zeta + 6x^2\ ,$$

$$S^2(3b-2\zeta) = b^2S^2(3) - 6b\zeta + \zeta^2 = 6b^2 - 6b\zeta + \zeta^2\ ,$$
$$\Lambda^2(2-3c) = \Lambda^2(2) - 6c + c^2\Lambda^2(-3) = 1 - 6c + 6c^2\ ,$$
$$\Lambda^2(2a+2b) = a^2\Lambda^2(2) + 4ab + b^2\Lambda^2(2) = a^2 + 4ab + b^2\ ,$$

and all these functions specialize to

$$P_2(x) = 3x^2/2 - x/2\ .$$

Exercises

Ex. 1.1. Let $\mathbb{A}$ be arbitrary, k be a positive integer. Express the product $S^k(1 - z\mathbb{A})\,\sigma_z(\mathbb{A})$ in the basis of Schur functions.

Solution. The product to decompose is made of two factors of different types. To make it more uniform, and easier to interpret, one introduces an alphabet $\mathbb{B}$ of cardinality k, such that $S^i(-\mathbb{B}) = S^i(-\mathbb{A})$, $i = 0, \ldots, k$. Then

$$S^k(1-z\mathbb{B})\,\sigma_z(\mathbb{A}) = \sigma_z(-\mathbb{B})\,\sigma_z(\mathbb{A}) = \sigma_z(\mathbb{A}-\mathbb{B}) = \sum z^n S^n(\mathbb{A}-\mathbb{B}) = \sum z^n S_{n;\,0^k}(\mathbb{A};\,\mathbb{B}) \ ,$$

using our favourite transformation of Schur functions.

The determinants $S_{n;\,0^k}(\mathbb{A};\,\mathbb{B})$ involve, as for concerns $\mathbb{B}$, only complete functions of degree $\le k$, but for those degrees, one can replace $\mathbb{B}$ by $\mathbb{A}$, and therefore the sum is

$$\sum z^n S_{n,0^k}(\mathbb{A}) = (-1)^k \sum_j z^j S_{1^k,\,j-k}(\mathbb{A})\,.$$

```
 Ex1:=proc(k,n)  local i,pol;
   pol:=1+convert([seq((-z)^i*e.i,i=1..k)],`+`);
   map(Tos,taylor(pol*(1+convert([seq(z^i*h.i,i=1..n)],`+`)),
                                          z, n+1),collect); end:
ACE>  Ex1(3,6);
   s[] -s[1,1,1,1]z^4 -s[2,1,1,1] z^5 -s[3,1,1,1] z^6 +s[]O(1)z^7
```

Ex. 1.2. Let $I = [i_1, \ldots, i_r]$ be a partition, n an integer : $n \ge i_r$. Check that

$$S_{I;\,0^n}(\mathbb{A},\mathbb{B}) = S_I(\mathbb{A} - \mathbb{B}) \ .$$

Solution. Since $\mathbb{B}$ is of arbitrary cardinality, one cannot directly apply the transformation lemma of multi-Schur functions given in the course.

```
#  incr=increasing partition, lA=list of alphabets
MultiSchur:=proc(incr,lA)   local n,i,j,ma;
 n:=nops(incr);
 transpose(array([seq(
  map(proc(k,i,ll) if (k>0) then s[k](op(i,ll)) elif (k=0)
  then 1 else 0 fi end,[seq(incr[i]-j+i,j=1..n)],i,lA),i=1..n)]));
 end;
ExSchur2:=proc(pa,n) local i,v;    # pa=decreasing partition
  v:=[seq(pa[nops(pa)-i],i=0..nops(pa)-1) ,0$n];
  MultiSchur(v, [A1$nops(pa), A2$n])
```

```
end:
ACE> aa:=ExSchur2([2,1], 3);
      [s[1](A1)   s[3](A1)    s[2](A2)    s[3](A2)    s[4](A2)]
      [   1       s[2](A1)    s[1](A2)    s[2](A2)    s[3](A2)]
aa := [   0       s[1](A1)       1        s[1](A2)    s[2](A2)]
      [   0          1           0           1        s[1](A2)]
      [   0          0           0           0           1    ]
ACE> SfAExpand(det(aa));
s[2,1](A1)-s[1](A2)s[1,1](A1)-s[1](A2)s[2](A1)+s[1](A1)s[1,1](A2)
      + s[1](A1) s[2](A2) - s[2,1](A2)
ACE> SfAExpand(s[2,1](A1-A2));
s[2,1](A1)-s[1](A2)s[1,1](A1)-s[1](A2)s[2](A1)+s[1](A1)s[1,1](A2)
      + s[1](A1) s[2](A2) - s[2,1](A2)
```

However, the Laplace expansion of the determinant $S_{I;0^n}(\mathbb{A},\mathbb{B})$ along the last n columns shows that, as a function of $\mathbb{B}$, it belongs to the linear span of the products $S^{j_1}(\mathbb{B})\cdots S^{j_n}(\mathbb{B})$, which is also the linear span of Schur functions $S_J(\mathbb{B})$, $J\in\mathbb{N}^n$. Similarly, $S_I(\mathbb{A}-\mathbb{B}) = \sum \pm S_{I/J^\sim}(\mathbb{A})S_J(\mathbb{B})$, with the same limit on the number of parts of J because $n\geq i_r$.

Now, a symmetric function which belongs to the linear span of S_J, $J\in\mathbb{N}^n$, is null (as a function of an arbitrary alphabet) iff it is null on alphabets of cardinality n.

We therefore have only to check the required identity for $\mathbb{B}$ of cardinality n, in which case it is given by subtracting $\mathbb{B}$ to the top rows of $S_{I;0^n}(\mathbb{A},\mathbb{B})$.

Ex. 1.3. Let $\mathbb{A}$ be arbitrary, x be a rank-1 element, $m,n\in\mathbb{N}$. Decompose the polynomial $S_{m^n}(\mathbb{A}+x)$ in the basis of polynomials $\{S^k(\mathbb{A}+x) : k=0,1,2,\ldots\}$ (all polynomials considered as polynomials in x with coefficients in $\mathfrak{Sym}(\mathbb{A})$).

Solution. Subtract x to all the rows, except the last one. The entries of the last row are complete functions of $\mathbb{A}+x$, and the other rows are independent of x. Therefore, the expansion of this new determinant along its last row is the required decomposition.

Ex. 1.4. Express the coefficients of the Taylor expansion (in z) of the rational function

$$\prod_{b\in\mathbb{B}}(1-zb)/\prod_{a\in\mathbb{A}}(1-za)$$

in terms of the coefficients of the numerator and denominator.

Solution. The function is $\sigma_z(\mathbb{A}-\mathbb{B}) = \sum z^n S^n(\mathbb{A}-\mathbb{B})$, but we want only to see the $S^i(-\mathbb{A})$, $S^j(-\mathbb{B})$. We have just to write

$$S^n(\mathbb{A}-\mathbb{B}) = (-1)^n S_{1^n}(-\mathbb{B}+\mathbb{A}) = (-1)^n S_{1^n;0}(-\mathbb{B};\ -\mathbb{A})\ ,$$

thanks to Ex. 1.2.

Ex. 1.5. Let $\mathbb{A}$ be arbitrary, x be rank-1 element, and $\mathbb{B}$ of cardinality 4 be such that

$$S_{4444}(\mathbb{A}+x) = S_{444}(\mathbb{A})\, S^4(\mathbb{B}+x)\ .$$

Show that $S_4(\mathbb{B}-\mathbb{A}) = 0$.
Solution. With the function $S_4(\mathbb{A}-\mathbb{B})$, one could have thought of writing it as $S_{4;0000}(\mathbb{A};\mathbb{B})$, but this is not the function that one has to study, and moreover the hypothesis that $\mathbb{B}$ be of cardinality 4 is not necessary as we shall see.

The hypothesis is that

$$x^4 S_{444}(\mathbb{A})+x^3 S_{1444}(\mathbb{A})+x^2 S_{2444}(\mathbb{A})+x S_{3444}(\mathbb{A})+S_{4444}(\mathbb{A}) = S_{4444}(\mathbb{A}+x)$$
$$= S_{444}(\mathbb{A})\, S^4(\mathbb{B}+x) = S_{444}(\mathbb{A})\Big(x^4+x^3 S_1(\mathbb{B})+x^2 S_2(\mathbb{B})+x S_3(\mathbb{B})+S_4(\mathbb{B})\Big)\ .$$

Expanding $S_4(\mathbb{B}-\mathbb{A}) = S_4(\mathbb{B}) - S_3(\mathbb{B})S_1(\mathbb{A}) + \cdots + S_{1111}(\mathbb{A})$, and replacing the complete functions of $\mathbb{B}$ by their values in terms of $\mathbb{A}$, one is asked to check the nullity of

$$S_{4444}(\mathbb{A})-S_{3444}(\mathbb{A})S_1(\mathbb{A})+S_{2444}(\mathbb{A})S_{11}(\mathbb{A})-S_{1444}(\mathbb{A})S_{111}(\mathbb{A})+S_{444}(\mathbb{A})S_{1111}(\mathbb{A})$$

but it easier to interpret this sum writing the Schur functions as determinants in $\Lambda^i(\mathbb{A})$ (this just involves taking conjugate partitions) :

$$\Lambda_{4444}(\mathbb{A})-\Lambda_{3444}(\mathbb{A})\Lambda_1(\mathbb{A})+\Lambda_{3344}(\mathbb{A})\Lambda_2(\mathbb{A})-\Lambda_{3334}(\mathbb{A})\Lambda_3(\mathbb{A})+\Lambda_{3333}(\mathbb{A})\Lambda_4(\mathbb{A})$$

Now, one sees that the sum is the Laplace expansion of the determinant $\Lambda_{44440}(\mathbb{A}) = 0$ along its last column.

Ex. 1.6. Given two finite alphabets $\mathbb{A},\mathbb{B}$, and a partition I, write $S_I(\mathbb{A}+\mathbb{B})\prod_{a\in\mathbb{A},b\in\mathbb{B}}(a-b)$ as a multi-Schur function.
Solution. Let $\mathbb{A}$, $\mathbb{B}$ be of respective cardinality α,β, and let $\square = \beta^\alpha$. Then, from Lemma (1.3.1), one gets

$$S_I(\mathbb{A}+\mathbb{B})\prod(a-b) = S_{I^\sim;\square}(-\mathbb{B};\mathbb{A}-\mathbb{B}) = S_{I^\sim;\square^\sim}(-\mathbb{A};\mathbb{B}-\mathbb{A})\ .$$

Ex. 1.7. Let $\mathbb{A}$ be of cardinality n, and $I, J \in \mathbb{N}^n$. Show that the determinant

$$\left|S_{j_k+k-h+i_{n-h+1}}(\mathbb{A})\right|$$

factorizes into $S_J(\mathbb{A})\, S_I(\mathbb{A})$. What can be said if n is not the cardinality of $\mathbb{A}$?

For example, for $n=3$, $I=[1,3,3]$, $J=[2,4,6]$, one has

$$\begin{array}{ccc|c} 0 & 1 & 2 & 3 \\ -1 & 0 & 1 & 3 \\ -2 & -1 & 0 & 1 \\ \hline 2 & 4 & 6 & \end{array} = \begin{vmatrix} S_5 & S_8 & S_{11} \\ S_4 & S_7 & S_{10} \\ S_1 & S_4 & S_7 \end{vmatrix} = S_{133}(\mathbb{A})\, S_{246}(\mathbb{A}) \ .$$

Solution.

```
ACE>  aa:= ProdSchur([2,6], [0,2]);
                                        [h4    h9]
                                        [h1    h6]
ACE>  factor(Toe_n(det(aa));
        e2^2  (- e2 + e1^2 ) (e1^4 - 3 e2 e1^2  + e2^2 )
  #  Preceding determinant is  s[ 84/20 ]. Take conjugate partitions
ACE> map(Toe_n, SfJtMat([[2$4,1$4],[1,1]], 'e'));
   [e1    e2    e4    e5    0     0     0     0 ]
   [1     e1    e3    e4    e5    0     0     0 ]
   [0     1     e2    e3    e4    e5    0     0 ]
   [0     0     e1    e2    e3    e4    e5    0 ]
   [0     0     0     1     e1    e2    e3    e4]
   [0     0     0     0     1     e1    e2    e3]
   [0     0     0     0     0     1     e1    e2]
   [0     0     0     0     0     0     1     e1]
```

Ex. 1.8. Compute the adjoint matrix of a Jacobi-Trudi matrix.
For example

```
ACE>       SfJtMat([5,4,1]), map(Tos, adj(SfJtMat([5,4,1])));
 [h5    h6    h7]   [     s[4,1]       -s[6, 1]        s[6,5]      ]
 [h3    h4    h5],  [-s[3,1]-s[4]  s[5,1] +s[6]  -s[5,5]-s[6,4]]
 [0     1     h1]   [      s[3]        -s[5]           s[5,4]    ]
```

Solution. The entries of the adjoint matrix are minors of the original matrix. In our case, they are minors of the Toeplitz matrix $\left[S^{j-i}\right]$, since the Jacobi-Trudi matrix itself is so.

The required entries are therefore skew Schur functions, up to sign. The example should be written

$$\begin{bmatrix} S_{56/11} & -S_{56/1} & S_{56} \\ S_{16/11} & -S_{16/1} & S_{16} \\ S_{14/11} & -S_{14/1} & S_{14} \end{bmatrix}$$

and this gives the general pattern.