Phase Structure
of Strongly Interacting Matter

J. Cleymans (Ed.)

Phase Structure of Strongly Interacting Matter

Proceedings of a Summer School on Theoretical Physics, Held at the University of Cape Town, South Africa, January 8–19, 1990

With 190 Figures

Springer-Verlag

Berlin Heidelberg New York London
Paris Tokyo Hong Kong Barcelona

Professor Dr. Jean Cleymans
Department of Physics, University of Cape Town, ZA-7700 Rondebosch, South Africa

Organizing Committee

J. Cleymans (Chairman)
C. A. Engelbrecht
H. G. Miller
R. D. Viollier

ISBN 978-3-642-87823-7 ISBN 978-3-642-87821-3 (eBook)
DOI 10.1007/978-3-642-87821-3

Softcover reprint of the hardcover 1st edition 1990

2155/3140-543210 – Printed on acid-free paper

Preface

The 6th Advanced Course in Theoretical Physics was held at the University of Cape Town, January 8–19, 1990. The topic of the course was "Phase Structure of Strongly Interacting Matter". There were ten invited speakers from overseas, each having up to six hours in which to present his field of research to a relatively small audience of about 50 participants. This allowed for the presentation of a broad, coherent and pedagogical review of the present status of the field. In addition there were several one-hour presentations by local participants.

The main emphasis of the course was on the study of the properties of high-density hot nuclear matter. This field is of particular interest because of the belief that a deconfined quark–gluon plasma could be created in such an environment when the temperature reaches about 200 MeV. In the nuclear regime a so-called "liquid-to-gas" phase transition is expected at a temperature of approximately 10–20 MeV. Both of these topics received ample attention at the school. Owing the nature of the field, there exists much overlapping interest from both the nuclear physics and high-energy particle physics communities. It is hoped that these proceedings will contribute to building a bridge between the two groups.

The course was made possible through generous contributions from:
- Foundation for Research Development (Pretoria)
- University of Cape Town and its Physics Department
- University of Pretoria and its Physics Department
- S.A. Network Computers

The financial support of Professors R.D. Viollier and H.G. Miller is also acknowledged. We furthermore acknowledge contributions from First National Bank and from Standard Bank.

Cape Town
May 1990

Jean Cleymans

Contents

Part I

Lectures
on Strongly Interacting Matter
and Its Phase Structure

Introduction to Perturbative QCD*

R.L. Thews

Physics Department, University of Arizona, Tucson, AZ85721, USA

1 Introduction

QCD (Quantum Chromodynamics) is a candidate for the theory of strong interactions, and has attained some status as a part of the so-called "Standard Model". I have restricted the scope of these introductory lectures from what is implied in the published title to perturbative aspects of the theory. Although this excludes most of the outstanding problems of the theory yet to be solved, what remains should provide the student with a well-defined and hopefully relevant set of calculations which may be compared with experiment. The level of these lectures is intended for students who have completed a graduate course in quantum mechanics including some relativistic material, and who have some familiarity with the Feynman diagram approach to perturbative calculations. The material has been taken from the following excellent reference books: "Gauge Theories of the Strong, Weak, and Electromagnetic Interactions", by Chris Quigg, Number 56 in the Frontiers in Physics series, Benjamin/Cummings, 1983; "Introduction to Gauge Field Theory", by D. Bailin and A. Love, Graduate Student Series in Physics, IOP Publishing, 1986; "Collider Physics", by Vernon D. Barger and Roger J. N.Phillips, Number 71 in the Frontiers in Physics series, Addison-Wesley, 1987; and "Applications of Perturbative QCD", by Richard D. Field, Number 77 in the Frontiers in Physics series, Addison-Wesley, 1989. Lists of excellent reviews and original references may be found in these books.

The starting point will be a summary of perturbative methods in QED, with emphasis on the standard notation and methods for free particle solutions and Feynman diagrams. The notions of conserved currents and local gauge invariance will be explored for the U(1) case. Then the natural extension to non-abelian gauge groups is shown to lead to self-interacting gauge fields. The specific color SU(3) group of QCD is then used to generate the Feynman rules for future use. The next subject is concerned with the "running coupling constant" of the theory, the very property which allows sensible perturbative calculations in some kinematic regions. Finally, some detailed calculations for application to electron-positron annihilation into hadrons are presented, with examples of regularization schemes for gluon emission processes.

* This work is supported by U.S. Department of Energy Grant DE-FG002-85ER40213

Phase Structure of Strongly Interacting Matter
Editor: J. Cleymans

2 QCD - Definitions and Basic Properties

QCD is a gauge theory, which means that the form of the interaction is specified by invariance of the physical equation under a group of transformations. To understand what this means, it is instructive to look at the classical Maxwell equations of electromagnetism. One can write the equations for the electric and magnetic fields $\vec{E}$ and $\vec{B}$ in terms of a vector and scalar potential $\vec{A}$ and ϕ. Since $\vec{\nabla} \cdot \vec{B} = 0$ one can write $\vec{B} = \vec{\nabla} \times \vec{A}$, and to satisfy $\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$, one can write $\vec{E} = -\vec{\nabla}\phi - \frac{\partial \vec{A}}{\partial t}$. There remains an ambiguity in specifying these potentials, which comes from the freedom to perform a so-called gauge transformation without changing the values of the physical $\vec{E}$ and $\vec{B}$ fields. We can introduce an arbitrary function $\Lambda(x, t)$ such that under a transformation

$$\vec{A} \rightarrow \vec{A} + \vec{\nabla}\Lambda \tag{1}$$

$$\phi \rightarrow \phi - \frac{\partial \Lambda}{\partial t} \tag{2}$$

no physical quantities are altered. (Note, however, in quantum mechanics phenomena such as the Aharanov-Bohm effect, in which $\vec{A}$ itself has a physical significance).

The Maxwell equation can be written in covariant notation in terms of the field-strength tensor

$$F^{\mu\nu} \equiv \partial^\nu A^\mu - \partial^\mu A^\nu \tag{3}$$

and the corresponding dual tensor

$$\tilde{F}^{\mu\nu} \equiv -\frac{1}{2}\epsilon^{\mu\nu\lambda\rho} F_{\lambda\rho} \tag{4}$$

where $A^\mu \equiv (\phi, \vec{A}), x^\mu \equiv (t, \vec{x})$ and $\partial_\mu \equiv \frac{\partial}{\partial x^\mu}$ transform as 4-vectors, $g^{\mu\nu} \equiv$ diag(1,-1,-1,-1) is the metric tensor, and $\epsilon^{\mu\nu\lambda\rho}$ is the totally antisymmetric tensor in 4 dimensions.

The homogeneous equations becomes $\partial_\mu \tilde{F}^{\mu\nu} = 0$, while the equations with source terms ρ and $\vec{J}$ are written

$$\partial_\mu F^{\mu\nu} = -J^\nu, \tag{5}$$

with

$$J^\nu \equiv (\rho, \vec{J}). \tag{6}$$

In this covariant notation, gauge freedom allows a choice for $\partial_\mu A^\mu$, which then gives an equation for the gauge function

$$\frac{\partial^2 \Lambda}{\partial t^2} - \nabla^2 \Lambda \equiv \Box \Lambda \tag{7}$$

In addition, because of the antisymmetry of $F^{\mu\nu}$, one has $\partial_\nu J^\nu = 0$, which is the continuity equation and indicates conservation of electric charge. The effect of the choice of gauge is readily seen in the field equation when written in the form

$$\partial_\mu F^{\mu\nu} = \partial_\mu(\partial^\nu A^\mu) - \partial_\mu \partial^\mu A^\nu = -J^\nu \tag{8}$$

or

$$\Box A^\nu = J^\nu + \partial^\nu(\partial_\mu A^\mu) \tag{9}$$

It is instructive to derive these equations from a Lagrangian density in a form suitable for use in a quantum field theory. The Lagrangian density is a function of fields and their derivatives $L(\varphi, \partial_\mu \varphi)$ such that the condition of zero variation of the action yields the Euler-Lagrange field equations of the theory.

$$\delta \int_{t_1}^{t_2} dt \int d^3x \mathcal{L} = 0 \Longrightarrow \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} = \frac{\partial \mathcal{L}}{\partial \varphi} \tag{10}$$

where one assumes that the fields and their derivatives are sufficienty localized in 3-space that surface integrals vanish at infinity. For example, consider a complex scalar field $\varphi(x,t)$ with Lagrangian

$$\mathcal{L} = |\partial_\mu \varphi|^2 - m^2 |\varphi|^2 \tag{11}$$

This yields the field equations

$$(\Box + m^2)\varphi = (\Box + m^2)\varphi^* = 0 \tag{12}$$

which indicate that momentum space solutions satisfy the usual $E^2 = p^2 + m^2$ energy-momentum relation for free particles.

For spin-$\frac{1}{2}$ particles, one can verify that the Lagrangian $L = \bar{\psi}(i\gamma^\mu \partial_\mu - m)\psi$ yields the Dirac equation $i\gamma^\mu \partial_\mu \psi = m\psi$ for the 4-component spinor $\psi(x)$, with corresponding equation for $\bar{\psi}(x) \equiv \psi^+(x)\gamma^o$. Finally, for the electromagnetic field, the Lagrangian $\mathcal{L} = -\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$ gives the zero-source Maxwell equations $\partial_\mu F^{\mu\nu} = 0$. Note that if one adds a mass term $\frac{1}{2}M^2 A^\mu A_\mu$ to get an equation appropriate for a massive vector field $\partial_\mu F^{\mu\nu} + M^2 A^\mu = 0$ the divergence of the equation implies a gauge choice $\partial_\mu A^\mu = 0$ which is required to eliminate the 4th degree of freedom from the vector field to properly decribe the three spin components. Note therefore that the mass term is not gauge invariant, as can be seen by direct substitution.

Now consider a case where the Lagrangian is invariant under some transformation of fields and/or coordinates. Then we can invoke Noether's Theorem, which indicates that this implies the existence of a conserved current j^μ, $(\partial_\mu j^\mu = 0)$ and a time-independent charge $Q \equiv \int d^3x j^o(t, \vec{x})$. This is simple to show if we restrict ourselves to transformations which affect the fields and their derivatives, but not the coordinates explicitly. Then

$$\delta S = 0 \Longrightarrow 0 = \int d^4x \Big[\frac{\partial \mathcal{L}}{\partial \varphi} - \partial_\mu \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\Big]\delta\varphi + \int d^4x \, \partial_\mu \Big[\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\delta\varphi\Big] \tag{13}$$

The first integrand is zero by the field equations. In the second, let $\delta\varphi = \chi\delta\lambda$ where λ is a parameter describing the infinitesimal transformation. Then the conserved current is read off directly

$$j^\mu \equiv \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)}\chi \tag{14}$$

As an example, consider the phase invariance in the scalar field case. For

$$\mathcal{L} = |\partial_\mu \varphi|^2 - m^2 |\varphi|^2 \tag{15}$$

let

$$\varphi \to e^{iq\lambda}\varphi \tag{16}$$

and

$$\varphi^* \rightarrow e^{-iq\lambda}\varphi^* \tag{17}$$

Then

$$\delta\varphi = iq\varphi\delta\lambda \Longrightarrow \chi_1 = iq\varphi \tag{18}$$

$$\delta\varphi^* = -iq\varphi^*\delta\lambda \Longrightarrow \chi_2 = -iq\varphi^* \tag{19}$$

and

$$j^\mu = \partial^\mu\varphi^* \cdot iq\varphi + \partial^\mu\varphi^* \cdot (-iq\varphi*) \tag{20}$$

$$\equiv -iq\varphi^*\overleftrightarrow{\partial^\mu}\varphi, \tag{21}$$

which can be identified with the electric current of the charged particle quanta of the complex scalar field. A similar quantity exists for the Dirac equation: L = $\bar{\psi}(i\gamma^\mu\partial_\mu - m)\psi$.

$$\psi \rightarrow e^{iq\lambda}\psi \Longrightarrow \chi_1 = iq\psi \tag{22}$$

$$\bar{\psi} \rightarrow \bar{\psi}e^{-iq\lambda}\psi \Longrightarrow \chi_2 = -iq\bar{\psi} \tag{23}$$

and since

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\bar{\psi})} = 0, \tag{24}$$

one obtains

$$j^\mu = -q\bar{\psi}\gamma^\mu\psi. \tag{25}$$

This type of transformation is called global, since the phase parameter is independent of space. One can also consider what are called local gauge transformations, in which the parameter $\lambda = \lambda(x)$. Obviously the Lagrangian densities we have considered so far are not invariant under such transformations because of the derivative terms $\partial_\mu\varphi$. One can insist on local gauge invariance by introducing what is called a covariant derivative which depends on a gauge field A_μ

$$D_\mu \equiv \partial_\mu + iqA_\mu \tag{26}$$

such that under the transformation $\varphi \rightarrow e^{iq\lambda(x)}\varphi$ one has $D_\mu\varphi \rightarrow e^{iq\lambda(x)}D_\mu\varphi$ which leaves the typical terms in the Lagrangian invariant. This restriction requires

$$(\partial_\mu + iqA'_\mu(x))e^{iq\lambda(x)}\varphi(x) = e^{iq\lambda(x)}(\partial_\mu + iq(A' + \partial_\mu\lambda))\varphi \tag{27}$$

which will work if the gauge field A^μ transforms as $A_\mu \rightarrow A_\mu - \partial_\mu\lambda(x)$. This local gauge invariance then specifies the form of an interaction between the gauge field and the original fields in the free-particle Lagrangian. For the Dirac Lagrangian L $= \bar{\psi}(i\gamma^\mu D_\mu - m)\psi$ one separates the free and interacting parts as L = $L_{free} - J^\mu A_\mu$, with $J^\mu = q\bar{\psi}\gamma^\mu\psi$. To obtain the QED Lagrangian, one only needs to add the free part of the electromagnetic field Lagrangian, since the $J^\mu A_\mu$ interacting part is common to both. Thus

$$\mathcal{L}_{QED} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \tag{28}$$

Note again that no photon mass term may be allowed since $A_\mu A^\mu$ cannot be made locally gauge invariant.

The phase invariance we have been looking at is actually invariance under the group of unitary transformations of fields U(1). The global invariance alone (λ = constant) implies a conserved current, while local invariance $\lambda = \lambda(x)$ constrains the form of an interaction between the Dirac fermions and the photon. One can extend this invariance principle to more complicated groups than U(1). QCD is such a theory, where the matter fields describe spin-$\frac{1}{2}$ quarks which occur in triplet of color transforming according to the group SU(3), and we insist that the Lagrangian be invariant under such transformations. A general transformation on the Dirac spinor ψ describing the quarks is

$$\psi(x) \to \psi'(x) = e^{-ig\Sigma_{a=1}^{8} T_a \Lambda_a(x)} \psi(x) \equiv U(x)\psi(x) \tag{29}$$

where $T_a, a = 1, ...8$ are the infintesimal generators of SU(3) (in this case represented by 3 x 3 matrices) which satisfy commutation relations

$$[T_a, T_b] = i f_{abc} T_c \tag{30}$$

where the f_{abc} are called the structure constants of the SU(3) algebra. To construct a Lagrangian which is invariant under this transformation, one introduces 8 vector gauge fields A_a^μ and the corresponding covariant derivatives

$$D^\mu \psi \equiv (\partial^\mu + igT_a A_a^\mu(x))\psi. \tag{31}$$

The invariance requirement

$$D^\mu \psi \to U(x) D^\mu \psi \tag{32}$$

tells us how the gauge fields must transform. Because the generators T_a do not commute (non-Abelian group) we get some additional terms.

$$(\partial^\mu + igT_a A_a^{\mu\prime})\psi' = U(x)(\partial^\mu \psi) + (\partial^\mu U)\psi + igT_a A_a^{\mu\prime}(U\psi) = U(\partial^\mu \psi) + igU(T_a A_a^\mu \psi). \tag{33}$$

Regard this as an operator equation acting on an arbitrary ψ, and multiply on the right by U^{-1}, to get

$$T_a A_a^{\mu\prime} = U[T_a A_a^\mu + i/gU^{-1}(\partial^\mu U)]U^{-1} \tag{34}$$

One then considers infintesimal transformations to show that this is satisfied if

$$A_a^{\mu\prime} = A_a^\mu + \partial^\mu \Lambda_a + g f_{abc} \Lambda_b A_c^\mu, \tag{35}$$

where the f_{abc} are the structure constants of the SU(3) algebra. The new non-abelian feature then brings in additional terms via the gauge field kinetic terms which indicate self-interactions of the gauge particles. To generate these kinetic terms, start with the usual field-strength tensor $F_a^{\mu\nu}$ for each gauge field $A_a^\mu(x)$ and form the matrix $F^{\mu\nu} \equiv F_a^{\mu\nu} T_a$. Consider a Lagrangian of the type as for the electromagnetic case.

$$\mathcal{L} = -\frac{1}{4} F_a^{\mu\nu} F_{a\mu\nu} = -\frac{1}{2} \mathrm{Trace}(F^{\mu\nu} F_{\mu\nu}) \tag{36}$$

where the last normalization comes from the fundamental representation of the generators

$$\mathrm{Trace}(T_a T_b) = \frac{1}{2}\delta_{ab} \tag{37}$$

We then need the transformation law $F' = UFU^{-1}$ to make L invariant, but explicit calculation shows that additional terms show up because of the non-abelian nature of the group. One can fix this by adding a commutator term to the field strength tensor

$$F^{\mu\nu} = \partial^\nu A^\mu - \partial^\mu A^\nu + ig[A_\nu, A_\mu] \tag{38}$$

where $A^\mu \equiv \sum_a A_a^\mu T_a$ which leads to

$$F_a^{\mu\nu} = \partial^\nu A_a^\mu - \partial^\mu A_a^\nu - gf_{abc}A_b^\nu A_c^\mu \tag{39}$$

and an SU(3) - invariant total QCD Lagrangian

$$\mathcal{L}_{QCD} = \bar{\psi}(i\gamma^\mu D_\mu - m)\psi - \frac{1}{4}F_a^{\mu\nu}F_{a\mu\nu} \tag{40}$$

The Euler-Lagrange equations then become

$$(i\gamma^\mu\partial_\mu - m)\psi = gT_a\gamma_\mu A_a^\mu\psi \tag{41}$$

and

$$\partial^\nu F_{\mu\nu}^a - gf_{abc}A_b^\nu F_{\mu\nu}^c = g\bar{\psi}\gamma_\mu T_a\psi \tag{42}$$

One is now in a position to generalize the Feynman diagram rules for perturbative graphs in QED to the corresponding manipulations in QCD.

2.1 External particles:

1. For external massless spin-1 bosons (photons) insert the polarization vector $\epsilon_\mu(\lambda)$ for $\lambda = \pm 1$, the physical helicity states. An explicit construction gives $\epsilon_\mu(\lambda = \pm 1) = \mp\frac{1}{\sqrt{2}}(0, 1, \pm i, 0)$ for momentum along the z-axis. They are both transverse, $\epsilon \cdot k = 0$, and normalized to $\epsilon_\mu\epsilon^\mu = -1$. For the massless gluons in QCD, the factors are exactly the same for each of the 8 colors. One then either sums or averages over color in the final or initial states, exactly as for spin.
2. For external spin-$\frac{1}{2}$ fermions (electrons, etc.) with momentum p and spin s, insert the free-particle solutions to the Dirac equation as follows: for a fermion in the initial state a factor $u(p, s)$ on the right; for a fermion in the final state a factor $\bar{u}(p, s)$ on the left; for an anti-fermion in the initial state a factor $\bar{v}(p, s)$ on the left; and for an anti-fermion in the final state a factor v(p,s) on the right. These solutions satisfy

$$\begin{aligned} (\not{p} - m)u(p, s) &= \bar{u}(p, s)(\not{p} - m) = 0; \\ \bar{u} &\equiv u^+\gamma^0 \end{aligned} \tag{43}$$

$$(\not{p} + m)v(p, s) = \bar{v}(p, s)(\not{p} + m) = 0 \tag{44}$$

with normalization

$$\sum_{spin} \bar{u}(p, s)u(p, s) = 2m \tag{45}$$

$$\sum_{spin} \bar{v}(p, s)v(p, s) = -2m \quad . \tag{46}$$

In spin sums for squared amplitudes, one encounters the projection operators

$$\sum_{spin} u(p,s)\bar{u}(p,s) = \not{p} + m \tag{47}$$

$$\sum_{spin} v(p,s)\bar{v}(p,s) = \not{p} - m. \tag{48}$$

In QCD, the spin-$\frac{1}{2}$ fermions are the quarks, and all factors are exactly the same for each of the 3 quark colors. Again one must sum or average over quark color in addition to spins.

2.2 Vertex terms:

In QED there is only one vertex corresponding to the coupling of a photon to a fermion. From the interaction term $q\bar{\psi}\gamma^{\mu}A_{\mu}\psi$ one reads off a factor $-ie\gamma^{\mu}$ for each vertex as in Figure 1. In QCD, there is a corresponding vertex for coupling of gluons to quarks. The interaction term is $g\bar{\psi}\gamma^{\mu}A_{a\mu}\psi$ from which one gets the factor $-ig\gamma^{\mu}(T_a)_{ij}$ for the vertex in Figure 2. Note that in this case we must label the quark $i,j = 1,2,3$ and the gluon $a = 1,\cdots,8$ colors. In addition, QCD has the gluon self-interacting terms. Figures 3 and 4 show these vertex functions. For the triple-gluon coupling which comes from the interaction term $gf_{abc}\partial^{\nu}A^{\mu}_{a}A^{b}_{\nu}A^{c}_{\mu}$, one gets a factor $-gf_{abc}F_{\lambda\mu\nu}(p_1,p_2,p_3)$, where $F_{\lambda\mu\nu}(p,k,q) \equiv (p-k)_{\nu}g_{\lambda\mu} + (k-q)_{\lambda}g_{\mu\nu} + (q-p)_{\mu}g_{\nu\lambda}$ and for the 4-gluon vertex interaction which comes from the square of the $gfAA$ term one inserts $-ig^2 f_{abe}f_{cde}(g_{\lambda\nu}g_{\mu\sigma} - g_{\lambda\sigma}g_{\mu\nu})$ plus two other permutations of pairs of gluons.

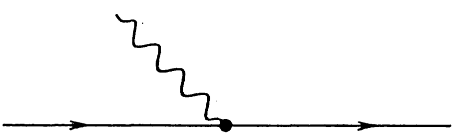


Figure 1: Vertex in QED.

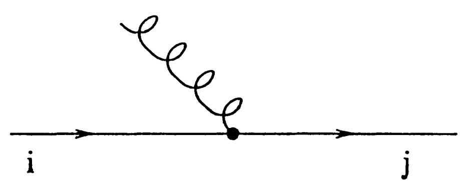


Figure 2: Vertex in QCD.

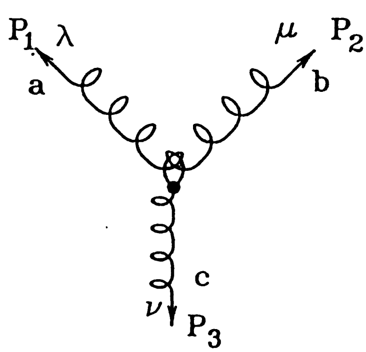


Figure 3: Triple gluon coupling.

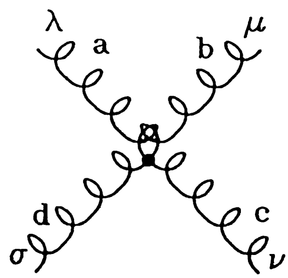


Figure 4: Four gluon coupling.

2.3 Internal lines:

1. For each internal fermion line in QED of four-momentum q, the propagator is

$$\frac{i(\not{q}+m)}{q^2-m^2+i\epsilon} \tag{49}$$

An internal anti-fermion is taken to be the equivalent fermion with $q \to -q$. In QCD, the propagator is the same, but with a multiplicative δ_{ij} where i and j are the color indices at either end of the line.

2. For each internal photon line in QED, one can write the propagator in Feynman gauge as

$$\frac{-ig_{\mu\nu}}{q^2+i\epsilon} \tag{50}$$

where μ and ν are the polarization vector indices at each end of the line. It is possible to use this form which in principle sums over unphysical polarization states of the photon, because gauge invariance insures that these unphysical polarization states will automatically cancel in all physical amplitudes. For QCD, a natural extension would involve just multiplying this propagator by a color delta function. This procedure however, does not give us the desired cancellation of unphysical polarization states in diagrams involving the triple-gluon vertex. One can see how this comes about by looking back at the field equations for the photon. The Feynman gauge is chosen by requiring $\partial_\mu A^\mu = 0$ in the field equation $\Box A^2 + \partial^\nu(\partial_\mu A^\mu) = J^\nu$, thus allowing the inversion of the equation to solve for A^ν in terms of J^ν. One could as easily have chosen a different gauge by inserting a gauge-fixing term in the Lagrangian, $-\frac{1}{2\xi}(\partial_\mu A^\mu)^2$, which leads to the field equations

$$\Box A^\nu - (1-\frac{1}{\xi})\partial^\nu(\partial_\mu A^\mu) = J^\nu \tag{51}$$

and the corresponding propagator

$$\frac{-i[g^{\mu\nu}+(\xi-1)q^\mu q^\nu/q^2]}{q^2+i\epsilon} \tag{52}$$

Note that the gauge-fixing term is itself gauge-invariant under $A_\mu \to A_\mu - \partial_\mu\lambda(x)$ as long as $\Box\lambda(x) = 0$. A similar construction for the QCD Lagrangian runs into trouble, since the non-abelian nature of the gauge group does not allow a simple constraint on the gauge transformation functions $\Lambda_a(x)$ to insure overall gauge invariance. One procedure is to choose a particular gauge to work in, at the expense of writing all expressions in a non-covariant manner. The other alternative is to add to the Lagrangian a color octet of ficticious scalar particles (Faddeev-Popov ghosts) which appear only in closed loops to cancel the unphysical gluon contributions. With the scalar field $\eta_a(x)$ one writes L $_{FP} = \partial_\mu\eta_a(\partial^\mu\eta_a + gf_{abc}\eta_b A^\mu_c)$. This produces a term for the scalar-scalar-gluon vertex, with a factor $gf_{abc}p_\mu$ with p_μ the ghost particle momentum, and also a propagator for the internal ghost line given by

$$\frac{-i\delta_{ab}}{q^2+i\epsilon} \tag{53}$$

2.4 Internal loops and indentical particles:

For each internal loop with momentum k, one performs the usual 4-dimension loop integral $\int d^4k/(2\pi)^4$. A factor of -1 is inserted for fermion loops and a factor of $1/n!$ for boson loops with n identical particles. In QCD the rules are the same, with the additional constraint that ghost loops, although they are bosons, also have the -1 factor as for fermions.

3 Running Coupling "Constants"

In the section we illustrate how interacting field theories can produce effective coupling strengths which vary with the scale of momentum. Calculational tools are introduced to first perform the calculation in QED. Then it is extended to QCD, where the non-abelian nature will manifest itself in producing an effective coupling which becomes very small at large momenta, or small distances. Thus one can find a kinematic region for some physical processes where an argument can be made for a sensible perturbative expansion.

For QED, we start by looking at the higher order terms in the photon propagator, due to virtual fermion loops, as in Figure 5.

$$P_{\mu\nu} = \frac{-ig_{\mu\nu}}{q^2} + -ig_{\mu\mu'} \cdot \prod_{\mu'\nu'} \cdot \frac{-ig_{\nu'\nu}}{q^2} \tag{54}$$

with

$$\prod_{\mu\nu} = -e^2 \int \frac{d^4k}{(2\pi)^4} \frac{\mathrm{Trace}[\gamma_\mu(\not{k} + m)\gamma_\nu(\not{k} - \not{q} + m)]}{(k^2 - m^2 + i\epsilon)((k-q)^2 - m^2 + i\epsilon)} \tag{55}$$

where the trace of γ-matrices comes from matching indices in the closed loop.

Note that the integral formally diverges as $k \to \infty$ (ultraviolet divergence). This is fixed by the usual renormalization procedure, where e is regarded as a "bare" charge, and the physical charge is relaterd to measurable quantities.

To evaluate the loop-integral, we use a parameterization of the propagators

$$\frac{i}{k^2 - m^2 + i\epsilon} = \int_0^\infty dx\ e^{ix(k^2 - m^2 + i\epsilon)} \tag{56}$$

to get

$$\prod_{\mu\nu} = e^2 \int \frac{d^4k}{(2\pi)^4} \int_0^\infty dx \int_0^\infty dy\ e^{ix(k^2-m^2+i\epsilon)+iy((k-q)^2-m^2+i\epsilon)} * Trace[\ ] \tag{57}$$

Then one can complete the square in the exponential by defining a new loop momentum $r_\mu \equiv k_\mu - \frac{y}{x+y} q_\mu$ and since the shift is finite $\int \frac{d^4k}{(2\pi)^4} \to \int \frac{d^4r}{(2\pi)^4}$, where we have

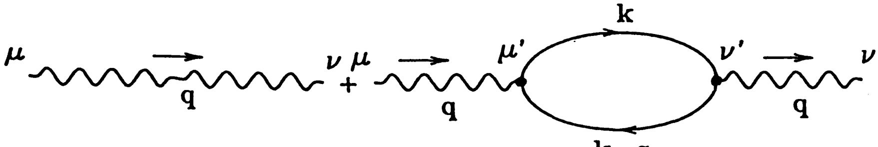


Figure 5: Lowest order corrections to the fermion propagator in QED.

anticipated a procedure which will make the integral finite. The resulting expression is

$$\begin{aligned}
\prod_{\mu\nu} &= 4e^2 \int_0^\infty dx \int_0^\infty dy \; e^{-i(x+y)(m^2-i\epsilon)} \; e^{\frac{ixy}{x+y}q^2} \\
& \int \frac{d^4r}{(2\pi)^2} e^{i(x+y)r^2} \Big[2r_\mu r_\nu - \frac{x-y}{x+y}(r_\mu q_\nu + q_\mu r_\nu) \\
&- \frac{2xy}{(x+y)^2} q_\mu q_\nu \\
&+ g_{\mu\nu}(m^2 - r^2 + \frac{x-y}{x+y} r \cdot q + \frac{xy}{(x+y)^2} q^2) \Big]
\end{aligned} \tag{58}$$

One can perform the r-integrals, using the identities

$$\int \frac{d^4r}{(2\pi)^4} e^{i\lambda r^2} = \frac{1}{16\pi^2 i\lambda^2} \tag{59}$$

and

$$\int \frac{d^4r}{(2\pi)^4} r^2 e^{i\lambda r^2} = \frac{1}{8\pi^2\lambda^3} \tag{60}$$

which comes directly from differentiation with respect to λ. The integrals involving single powers of r_μ momentum vanish identically due to the symmetric integration region, as can be seen by the substitution $r_\mu \to -r_\mu$. The integrals involving $r_\mu r_\nu$ can then be replaced by $\frac{1}{4}g_{\mu\nu}r^2$. The resulting terms in the integrand can then be grouped as follows

$$(g_{\mu\nu}q^2 - q_\mu q_\nu)\frac{2xy}{(x+y)^4} + \frac{g_{\mu\nu}}{(z+y)^2}\left[m^2 - \frac{i}{x+y} - \frac{xyq^2}{(x+y)^2}\right] \tag{61}$$

where the first combination has the correct momentum dependence for the propagator to insure current conservation $q^\mu \prod_{\mu\nu} = q^\nu \prod_{\mu\nu} = 0$. Hence the remaining term proportional to $g_{\mu\nu}$ which violates this property must vanish, as can be seen by direct computation. The nonvanishing coefficient of $(g_{\mu\nu}q^2 - q_\mu q_\nu)$ is then

$$\triangle(q^2, m^2) = -\frac{2i\alpha}{\pi}\int_0^\infty dx \int_0^\infty dy \; \frac{xy}{(x+y)^4} e^{-i(x+y)(m^2-i\epsilon)} e^{\frac{ixy}{x+y}q^2} \tag{62}$$

where we have defined the usual fine structure constant $\alpha \equiv \frac{e^2}{4\pi}$ and the ultraviolet divergence is seen to be present as a log-type singularity in the $x = y = 0$ region. One can use the Pauli-Villars method of regularization in this case. One inserts negative counterterms in the Lagrangian with $m^2 \to \Lambda^2 \gg q^2, m^2$, and the take the limit $\Lambda \to \infty$ for physical quantities. To proceed furthe r, it is convenient to insert a factor of unity in the integrand, written as

$$f(x,y) = \int_0^\infty \delta(\lambda - x - y) d\lambda \cdot f(x,y) \tag{63}$$

and then rescale x and y with a factor of λ to obtain

$$\int_0^\infty \frac{d\lambda}{\lambda}\delta(1 - x - y) f(\lambda x, \lambda y) \tag{64}$$

so that the upper limits on x and y are 1. Then the delta-function is used to perform the y-integral, to obtain

$$\triangle(q^2, m^2) = -\frac{2i\alpha}{\pi}\int_0^1 dxx(1-x)\int_0^\infty \frac{d\lambda}{\lambda} e^{-i\lambda(m^2-i\epsilon)} e^{i\lambda x(1-x)q^2} \tag{65}$$

With the subtraction for regularization, one then gets

$$\bar{\Delta} \equiv \Delta(q^2, m^2) - \Delta(q^2, \Lambda) = \frac{2i\alpha}{\pi}\int_0^1 dxx(1-x)\log\left(\frac{\Lambda^2}{m^2 - q^2x(1-x)}\right) \tag{66}$$

for $\Lambda \to \infty$ and $q^2 < 4m^2$.
In scattering reactions the conserved current coupling insures that the $q_\mu q_\nu$ does not contribute, so that the effective one-loop propagator is

$$-\frac{ig_{\mu\nu}}{q^2}\left[1 - \frac{\alpha}{3\pi}\log\left(\frac{\Lambda^2}{m^2}\right) + \frac{2\alpha}{\pi}\int_0^1 dxx(1-x)\log\left(1 - \frac{q^2x(1-x)}{m^2}\right)\right] \tag{67}$$

As $q^2 \to 0$ one can absorb the $1 - \frac{\alpha}{3\pi}\log(\frac{\Lambda^2}{m^2})$ factor into the coupling to renormalize the charge, and then let $\Lambda \to \infty$. In general, the effective coupling for one loop corrections takes the form

$$\alpha_{eff}(q^2) = \alpha(1 + \alpha B(q^2)) \tag{68}$$

where $B(q^2)$ is a divergent quantity. The most divergent graphs are products of these factors, coming from successive insertions of fermion loops in the propagator. One can formally sum the geometric series which results to get

$$\alpha_{eff}(q^2) = \alpha(1 + \alpha B(q^2) + (\alpha B(q^2))^2 + \ldots) = \frac{\alpha}{1 - \alpha B(q^2)}$$

It is instructive to write this as

$$\frac{1}{\alpha_{eff}(q^2)} = \frac{1}{\alpha} - B(q^2) \tag{69}$$

and to then define the renormalization point in QED at $q^2 = 0$, where $\alpha_R \equiv \alpha_{eff}(q^2 = 0)$ is measured to be approximately 1/137 from the Coulomb force law. Then one can write the effective coupling at any other momentum scale as

$$\frac{1}{\alpha_{eff}(q^2)} = \frac{1}{\alpha_R} + B(0) - B(q^2) \tag{70}$$

where no divergent terms appear in the difference

$$B(0) - B(q^2) = -\frac{2}{\pi}\int_0^1 dxx(1-x)\log\left(1 - \frac{q^2x(1-x)}{m^2}\right) \tag{71}$$

At large values of $Q^2 \equiv -q^2 > 0$, one obtains the effective coupling

$$\alpha_{eff}(Q^2) = \frac{\alpha_R}{1 - \alpha_R/3\pi\log(Q^2/m^2)} \tag{72}$$

This has the interesting property that as Q^2 increases (small distances) the effective coupling increases. One can interpret this effect in terms of the polarization of the QED vacuum by virtual e^+e^- pairs which lead to screening of the charge at large distances. As one probes smaller and smaller distances, however, the screening effect i s reduced and the effective coupling becomes larger. In fact, no matter how small the coupling at the renormalization point, one can find some large Q^2 for which the coupling becomes large and perturbative calculations no longer make sense. In practice, however, the smallness of m^2 and α_R make these Q^2 values very large. For example, to see an increase of a factor of 2 in the effective coupling requires

$$\begin{aligned} Q^2 &= m^2 e^{\frac{3\pi}{2\alpha_R}} \\ &\approx 10^{273}\ \mathrm{GeV}^2 \quad ! \end{aligned} \tag{73}$$

One can now perform the same calculation for QCD. For the quark loop correction to the gluon propagator, one can take over the QED calculation directly, with the substitution $-ie\gamma^\mu \rightarrow -ig\gamma^\mu T_a^{ij}$, such that $\alpha = \frac{e^2}{4\pi}$ is replaced by $\alpha_s \mathrm{Trace}(T_a T_b) = \frac{1}{2}\alpha_s \delta_{ab}$, where $\alpha_s = \frac{g^2}{4\pi}$ plays the role of the QCD coupling strength. The renormalization point is taken at a spacelike point $q^2 = -\mu^2$, and we set the quark masses to zero. The resulting propagator correction is, for $Q^2 = -q^2 \gg \mu^2$,

$$\bar{\prod}^{Quarks}_{\mu\nu,ab} = -i\frac{\alpha_s}{4\pi}\delta_{ab}(q_\mu q_\nu - q^2 g_{\mu\nu}) \cdot \frac{4}{3}\log(\frac{Q^2}{\mu^2}) \cdot \eta_f \tag{74}$$

where the factor of η_f sums over the number of flavors of quarks. There are two entirely gluonic contributions to $\bar{\prod}_{\mu\nu,ab}$. The first comes from the 4-gluon coupling term involving the diagram in Figure 6a, and can be shown to be zero when dimensional regularization is used (Pauli-Villars will not work for closed gluon loops - we introduce the technicalities of dimensional regularization in the next section). The second diagram is nonzero, coming from the gluon loop involving two triple-gluon couplings (Figure 6b). The corresponding calculation gives, in Landau gauge ($\xi = 0$)

$$\begin{aligned} \bar{\prod}^{Gluons}_{\mu\nu,ab} &= i\frac{\alpha_s}{4\pi} f^{acd} f^{bcd} \log(\frac{Q^2}{\mu^2}) \\ &* \ (\frac{11}{6} q_\mu q_\nu - \frac{19}{12} q^2 g_{\mu\nu} + \frac{1}{2}(q_\mu q_\nu - q^2 g_{\mu\nu})) \end{aligned} \tag{75}$$

This contribution by itself does not satisfy the current conservation relations

$$q^\mu \prod_{\mu\nu,ab} = 0 \tag{76}$$

and it is at this point where the need for the ghost particle contribution becomes evident. The relevant diagram is a ghost loop in the gluon propagator (Figure 6c),

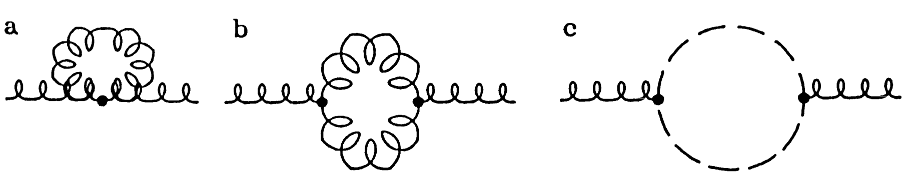


Figure 6: Lowest order corrections to the gluon propagator in QCD.

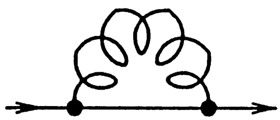
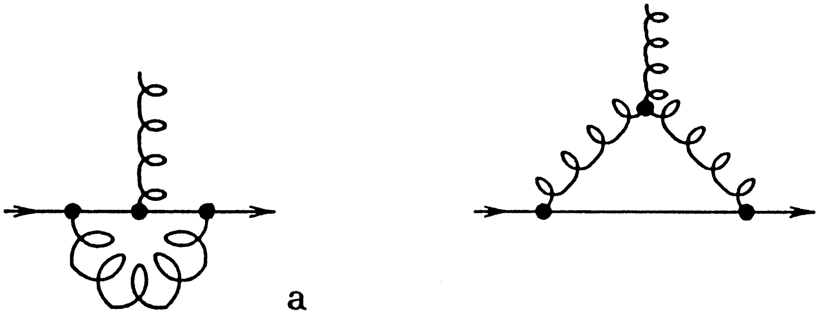


Figure 7: Correction to the fermion propagator.



a

b

Figure 8: Corrections to the vertex in QCD.

and gives

$$\overline{\prod}^{Ghost}_{\mu\nu,ab} = -i\frac{\alpha_s}{4\pi}f^{acd}f^{bcd}\log(\frac{Q^2}{\mu^2})\cdot(\frac{1}{6}q_\mu q_\nu + \frac{1}{12}q^2 g_{\mu\nu}) \qquad (77)$$

One can perform the sum $f^{acd}f^{bcd} = N\delta_{ab}$ with N = number of colors (3), so that the sum of all three contribution gives a transverse (but gauge-dependent) result for $\xi = 0$:

$$\overline{\prod}^{Total}_{\mu\nu,ab} = i\frac{\alpha_s}{4\pi}\delta_{ab}(q_\mu q_\nu - q^2 g_{\mu\nu})\log(\frac{Q^2}{\mu^2})\cdot(\frac{13}{6}N - \frac{2}{3}\eta_f) \qquad (78)$$

To get the total contribution to the effective coupling, one must consider all other order α_s loop corrections to a gluon exchange diagram between quarks. The modification to the external quark lines from a single gluon (Figure 7) in this order can be shown to vanish in Landau gauge. The vertex modification comes in two parts, from diagrams in Figure 8.

One obtains (again in Landau gauge) a modification term

$$\triangle\Gamma^\mu_a = -igT_a\gamma^\mu\cdot(-\frac{\alpha_s}{4\pi})\log(\frac{Q^2}{\mu^2})\cdot\frac{3}{4}N \qquad (79)$$

where the factor of N comes from

$$f^{abc}T^bT^c = \frac{i}{2}NT^a \qquad (80)$$

Thus this term comes entirely from the triple-gluon vertex and hence was absent in the QED calculation. To put everything together, one adds one correction for the gluon propagator to two of the vertex correction factors, one on each vertex of the basic gluon exchange diagram, to get the expected

$$\alpha_s(Q^2) = \alpha(1 + \alpha B(Q^2)) \qquad (81)$$

with