Variational Principles in Mathematical Physics, Geometry, and Economics

Qualitative Analysis of Nonlinear Equations and Unilateral Problems

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Contents

Preface

> For since the fabric of the universe is most perfect and the work of a most wise Creator, nothing at all takes place in the universe in which some rule of maximum or minimum does not appear.
>
> Leonhard Euler (1707–1783)

The roots of the calculus of variations go back to the 17th century. Indeed, Johann Bernoulli raised as a challenge the "Brachistochrone Problem" in 1696. The same year, when he heard of this problem, Sir Isaac Newton found that he could not sleep until he had solved it. Having done so, he published the solution anonymously. Bernoulli, however, knew at once that the author of the solution was Newton and, cf. [291], in a famous remark asserted that he "recognized the Lion by the print of its paw".

However, the modern calculus of variations appeared in the middle of the 19th century, as a basic tool in the qualitative analysis of models arising in physics. Indeed, *it was Riemann who aroused great interest in them [problems of the calculus of variations] by proving many interesting results in function theory by assuming Dirichlet's principle* (Charles B. Morrey Jr., [219]). The characterization of phenomena by means of variational principles has been a cornerstone in the transition from classical to contemporary physics. Since the middle part of the twentieth century, the use of variational principles has developed into a range of tools for the study of nonlinear partial differential equations and many problems arising in applications. Cf. Ioffe and Tikhomirov [143], *the term "variational principle" refers essentially to a group of re-*

sults showing that a lower semi-continuous, lower bounded function on a complete metric space possesses arbitrarily small perturbations such that the perturbed function will have an absolute (and even strict) minimum.

This monograph is an original attempt to develop the modern theory of the calculus of variations from the points of view of several disciplines. This theory is one of the twin pillars on which nonlinear functional analysis is built. The authors of this volume are fully aware of the limited achievements of this volume as compared with the task of understanding the force of variational principles in the description of many processes arising in various applications. Even though necessarily limited, the results in this book benefit from many years of work by the authors and from interdisciplinary exchanges between them and other researchers in this field.

One of the main objectives of this book is to let physicists, geometers, engineers, and economists know about some basic mathematical tools from which they might benefit. We would also like to help mathematicians learn what applied calculus of variations is about, so that they can focus their research on problems of real interest to physics, economics, engineering, as well as geometry or other fields of mathematics. We have tried to make the mathematical part accessible to the physicist and economist, and the physical part accessible to the mathematician, without sacrificing rigor in either case. The mathematical technicalities are kept to a minimum within the book, enabling the discussion to be understood by a broad audience. Each problem we develop in this book has its own difficulties. That is why we intend to develop some standard and appropriate methods that are useful and that can be extended to other problems. However, we do our best to restrict the prerequisites to the essential knowledge. We define as few concepts as possible and give only basic theorems that are useful for our topic. The authors use a first-principles approach, developing only the minimum background necessary to justify mathematical concepts and placing mathematical developments in context. The only prerequisite for this volume is a standard graduate course in partial differential equations, drawing especially from linear elliptic equations to elementary variational methods, with a special emphasis on the maximum principle (weak and strong variants). This volume may be used for self-study by advanced graduate students and as a valuable reference for researchers in pure and applied mathematics and related fields. Nevertheless, both the presentation style and the choice of the material make the present book accessible to all newcomers to this modern research field which lies at the interface between pure and applied mathematics.

Each chapter gives full details of the mathematical proofs and subtleties. The book also contains many exercises, some included to clarify simple points of exposition, others to introduce new ideas and techniques, and a few containing relatively deep mathematical results. Each chapter concludes with historical notes. Five appendices illustrate some basic mathematical tools applied in this book: elements of convex analysis, function spaces, category and genus, Clarke and Degiovanni gradients, and elements of set–valued analysis. These auxiliary chapters deal with some analytical methods used in this volume, but also include some complements. This unique presentation should ensure a volume of interest to mathematicians, engineers, economists, and physicists. Although the text is geared toward graduate students at a variety of levels, many of the book's applications will be of interest even to experts in the field.

We are very grateful to Diana Gillooly, Editor for Mathematics, for her efficient and enthusiastic help, as well as for numerous suggestions related to previous versions of this book. Our special thanks go also to Clare Dennison, Assistant Editor for Mathematics and Computer Science, and to the other members of the editorial technical staff of Cambridge University Press for the excellent quality of their work.

Our vision throughout this volume is closely inspired by the following prophetic words of Henri Poincaré on the role of partial differential equations in the development of other fields of mathematics and in applications: *A wide variety of physically significant problems arising in very different areas (such as electricity, hydrodynamics, heat, magnetism, optics, elasticity, etc...) have a family resemblance and should be treated by common methods.* (Henri Poincaré, [245]).

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P A R T I

Variational Principles in Mathematical Physics

1
Variational Principles

> A man is like a fraction whose numerator is what he is and whose denominator is what he thinks of himself. The larger the denominator the smaller the fraction.
>
> Leo Tolstoy (1828–1910)

Variational principles are very powerful techniques at the interplay between nonlinear analysis, calculus of variations, and mathematical physics. They have been inspired and have important applications in modern research fields such as geometrical analysis, constructive quantum field theory, gauge theory, superconductivity, etc.

In this chapter we shortly recall the main variational principles which will be used in the sequel, as Ekeland and Borwein-Preiss variational principles, minimax- and minimization-type principles (mountain pass theorem, Ricceri-type multiplicity theorems, Brézis-Nirenberg minimization technique), the principle of symmetric criticality for non-smooth Szulkin-type functionals, as well as the Pohozaev's fibering method.

1.1 Minimization techniques and Ekeland variational principle

Many phenomena arising in applications (such as geodesics or minimal surfaces) can be understood in terms of the minimization of an energy functional over an appropriate class of objects. For the problems of mathematical physics, phase transitions, elastic instability, and diffraction of the light are among the phenomena that can be studied from this point of view.

A central problem in many nonlinear phenomena is if a bounded from below and lower semi-continuous functional f attains its infimum. A simple function when the above statement clearly fails is $f : \mathbb{R} \to \mathbb{R}$ defined by $f(s) = e^{-s}$. Nevertheless, further assumptions either on f or on its domain may give a satisfactory answer. In the sequel, we give two useful forms of the well-known Weierstrass theorem.

Theorem 1.1 [Minimization; compact case] *Let X be a compact topological space and $f : X \to \mathbb{R}\cup\{+\infty\}$ a lower semi-continuous functional. Then f is bounded from below and its infimum is attained on X.*

Proof The set X can be covered by the open family of sets $S_n := \{u \in X : f(u) > -n\}$, $n \in \mathbb{N}$. Since X is compact, there exists a finite number of sets $S_{n_0}, \ldots, S_{n_l}$ which also cover X. Consequently, $f(u) > -\max\{n_0, \ldots, n_l\}$ for all $u \in X$.

Let $s = \inf_X f > -\infty$. Arguing by contradiction, we assume that s is not achieved which means in particular that $X = \cup_{n=1}^{\infty}\{u \in X : f(u) > s+1/n\}$. Due to the compactness of X, there exists a number $n_0 \in \mathbb{N}$ such that $X = \cup_{n=1}^{n_0}\{u \in X : f(u) > s + 1/n\}$. In particular, $f(u) > s + 1/n_1$ for all $u \in X$ which is in contradiction with $s = \inf_X f > -\infty$. □

The following result is a very useful tool in the study of various partial differential equations where no compactness is assumed on the domain of the functional.

Theorem 1.2 [Minimization; noncompact case] *Let X be a reflexive Banach space, M be a weakly closed, bounded subset of X, and $f : M \to \mathbb{R}$ be a sequentially weak lower semi-continuous function. Then f is bounded from below and its infimum is attained on M.*

Proof We argue by contradiction, that is, we assume that f is not bounded from below on M. Then, for every $n \in \mathbb{N}$ there exists $u_n \in M$ such that $f(u_n) < -n$. Since M is bounded, the sequence $\{u_n\} \subset M$ is so. Due to the reflexivity of X, one may subtract a subsequence $\{u_{n_k}\}$ of $\{u_n\}$ which weakly converges to an element $\tilde{x} \in X$. Since M is weakly closed, $\tilde{x} \in M$. Since $f : M \to \mathbb{R}$ is sequentially weak lower semi-continuous, we obtain that $f(\tilde{x}) \leq \liminf_{k\to\infty} f(u_{n_k}) = -\infty$, contradiction. Therefore, f is bounded from below.

Let $\{u_n\} \subset M$ be a minimizing sequence of f over M, that is, $\lim_{n\to\infty} f(u_n) = \inf_M f > -\infty$. As before, there is a subsequence

$\{u_{n_k}\}$ of $\{u_n\}$ which weakly converges to an element $\overline{x} \in M$. Due to the sequentially weak lower semi-continuity of f, we have that $f(\overline{x}) \leq \liminf_{k\to\infty} f(u_{n_k}) = \inf_M f$, which concludes the proof. □

For any bounded from below, lower semi-continuous functional f, Ekeland's variational principle provides a minimizing sequence whose elements minimize an appropriate sequence of perturbations of f which converges locally uniformly to f. Roughly speaking, Ekeland's variational principle states that there exist points which are almost points of minima and where the "gradient" is small. In particular, it is not always possible to minimize a nonnegative continuous function on a complete metric space. Ekeland's variational principle is a very basic tool that is effective in numerous situations, which led to many new results and strengthened a series of known results in various fields of analysis, geometry, the Hamilton-Jacobi theory, extremal problems, the Ljusternik-Schnirelmann theory, etc.

Its precise statement is as follows.

Theorem 1.3 [Ekeland's variational principle] *Let (X, d) be a complete metric space and let $f : X \to \mathbb{R} \cup \{\infty\}$ be a lower semi-continuous, bounded from below functional with $D(f) = \{u \in X : f(u) < \infty\} \neq \emptyset$. Then for every $\varepsilon > 0$, $\lambda > 0$, and $u \in X$ such that*

$$f(u) \leq \inf_X f + \varepsilon$$

there exists an element $v \in X$ such that

a) $f(v) \leq f(u)$;
b) $d(v, u) \leq \frac{1}{\lambda}$;
c) $f(w) > f(v) - \varepsilon\lambda d(w, v)$ *for each* $w \in X \setminus \{v\}$.

Proof It is sufficient to prove our assertion for $\lambda = 1$. The general case is obtained by replacing d by an equivalent metric λd. We define the relation on X:

$$w \leq v \iff f(w) + \varepsilon d(v, w) \leq f(v).$$

It is easy to see that this relation defines a partial ordering on X. We now construct inductively a sequence $\{u_n\} \subset X$ as follows: $u_0 = u$, and assuming that u_n has been defined, we set

$$S_n = \{w \in X : w \leq u_n\}$$

and choose $u_{n+1} \in S_n$ so that

$$f(u_{n+1}) \leq \inf_{S_n} f + \frac{1}{n+1}.$$

Since $u_{n+1} \leq u_n$ then $S_{n+1} \subset S_n$ and by the lower semi-continuity of f, S_n is closed. We now show that $\text{diam} S_n \to 0$. Indeed, if $w \in S_{n+1}$, then $w \leq u_{n+1} \leq u_n$ and consequently

$$\varepsilon d(w, u_{n+1}) \leq f(u_{n+1}) - f(w) \leq \inf_{S_n} f + \frac{1}{n+1} - \inf_{S_n} f = \frac{1}{n+1}.$$

This estimate implies that

$$\text{diam} S_{n+1} \leq \frac{2}{\varepsilon(n+1)}$$

and our claim follows. The fact that X is complete implies that $\bigcap_{n \geq 0} S_n = \{v\}$ for some $v \in X$. In particular, $v \in S_0$, that is, $v \leq u_0 = u$ and hence

$$f(v) \leq f(u) - \varepsilon d(u, v) \leq f(u)$$

and moreover

$$d(u, v) \leq \frac{1}{\varepsilon}(f(u) - f(v)) \leq \frac{1}{\varepsilon}(\inf_X f + \varepsilon - \inf_X f) = 1.$$

Now, let $w \neq v$. To complete the proof we must show that $w \leq v$ implies $w = v$. If $w \leq v$, then $w \leq u_n$ for each integer $n \geq 0$, that is $w \in \cap_{n \geq 0} S_n = \{v\}$. So, $w \not\leq v$, which is actually c). □

In $\mathbb{R}^N$ with the Euclidean metric, properties a) and c) in the statement of Ekeland's variational principle are completely intuitive as Figure 1.1 shows. Indeed, assuming that $\lambda = 1$, let us consider a cone lying below the graph of f, with slope $+1$, and vertex projecting onto u. We move up this cone until it first touches the graph of f at some point $(v, f(v))$. Then the point v satisfies both a) and c).

In the particular case $X = \mathbb{R}^N$ we can give the following simple alternative proof to Ekeland's variational principle, due to Hiriart-Urruty, [139]. Indeed, consider the perturbed functional

$$g(w) := f(w) + \varepsilon\lambda \|w - u\|, \qquad w \in \mathbb{R}^N.$$

Since f is lower semi-continuous and bounded from below, then g is lower semi-continuous and $\lim_{\|w\| \to \infty} g(w) = +\infty$. Therefore there exists $v \in \mathbb{R}^N$ minimizing g on $\mathbb{R}^N$ such that for all $w \in \mathbb{R}^N$

$$f(v) + \varepsilon\lambda \|v - u\| \leq f(w) + \varepsilon\lambda \|w - u\|. \tag{1.1}$$

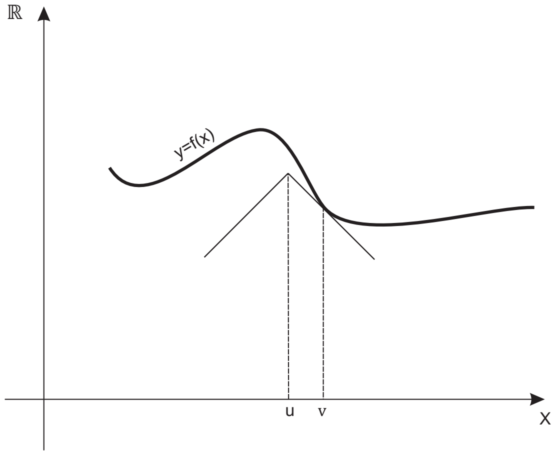

Fig. 1.1. Geometric illustration of Ekeland's variational principle.

By letting $w = u$ we find

$$f(v) + \varepsilon\lambda \|v - u\| \leq f(u)$$

and a) follows. Now, since $f(u) \leq \inf_{\mathbb{R}^N} f + \varepsilon$, we also deduce that $\|v - u\| \leq 1/\lambda$.

We infer from relation (1.1) that for any w,

$$f(v) \leq f(w) + \varepsilon\lambda \left[\|w - u\| - \|v - u\|\right] \leq f(w) + \varepsilon\lambda \|w - u\| ,$$

which is the desired inequality c).

Taking $\lambda = \dfrac{1}{\sqrt{\varepsilon}}$ in the above theorem we obtain the following property.

Corollary 1.1 *Let (X, d) be a complete metric space and let $f : X \to \mathbb{R} \cup \{\infty\}$ be a lower semi-continuous, bounded from below and $D(f) = \{u \in X : f(u) < \infty\} \neq \emptyset$. Then for every $\varepsilon > 0$ and every $u \in X$ such that*

$$f(u) \leq \inf_X f + \varepsilon$$

there exists an element $u_\varepsilon \in X$ such that

a) $f(u_\varepsilon) \leq f(u)$;
b) $d(u_\varepsilon, u) \leq \sqrt{\varepsilon}$;
c) $f(w) > f(u_\varepsilon) - \sqrt{\varepsilon}\, d(w, u_\varepsilon)$ *for each* $w \in X \setminus \{u_\varepsilon\}$.

Let $(X, \|\cdot\|)$ be a real Banach space, $X^\star$ its topological dual endowed with its natural norm, denoted for simplicity also by $\|\cdot\|$. We denote by $\langle \cdot, \cdot \rangle$ the duality mapping between X and $X^\star$, that is, $\langle x^\star, u \rangle = x^\star(u)$ for every $x^\star \in X^\star, u \in X$. Theorem 1.3 readily implies the following property, which asserts the existence of *almost critical points*. In order words, Ekeland's variational principle can be viewed as a generalization of the Fermat theorem which establishes that interior extrema points of a smooth functional are, necessarily, critical points of this functional.

Corollary 1.2 *Let X be a Banach space and let $f : X \to \mathbb{R}$ be a lower semi-continuous functional which is bounded from below. Assume that f is Gâteaux differentiable at every point of X. Then for every $\varepsilon > 0$ there exists an element $u_\varepsilon \in X$ such that*

(i) $f(u_\varepsilon) \leq \inf_X f + \varepsilon$;

(ii) $\|f'(u_\varepsilon)\| \leq \varepsilon$.

Letting $\varepsilon = 1/n$, $n \in \mathbb{N}$, Corollary 1.2 gives rise a minimizing sequence for the infimum of a given function which is bounded from below. Note however that such a sequence need not converge to any point. Indeed, let $f : \mathbb{R} \to \mathbb{R}$ defined by $f(s) = e^{-s}$. Then, $\inf_{\mathbb{R}} f = 0$, and any minimizing sequence fulfilling (a) and (b) from Corollary 1.2 tends to $+\infty$. The following definition is dedicated to handle such situations.

Definition 1.1 (a) A function $f \in C^1(X, \mathbb{R})$ satisfies the Palais-Smale condition at level $c \in \mathbb{R}$ (shortly, $(PS)_c$-condition) if every sequence $\{u_n\} \subset X$ such that $\lim_{n\to\infty} f(u_n) = c$ and $\lim_{n\to\infty} \|f'(u_n)\| = 0$, possesses a convergent subsequence.

(b) A function $f \in C^1(X, \mathbb{R})$ satisfies the Palais-Smale condition (shortly, (PS)-condition) if it satisfies the Palais-Smale condition at every level $c \in \mathbb{R}$.

Combining this compactness condition with Corollary 1.2, we obtain the following result.

Theorem 1.4 *Let X be a Banach space and a function $f \in C^1(X, \mathbb{R})$ which is bounded from below. If f satisfies the $(PS)_c$-condition at level $c = \inf_X f$, then c is a critical value of f, that is, there exists a point $u_0 \in X$ such that $f(u_0) = c$ and u_0 is a critical point of f, that is, $f'(u_0) = 0$.*

1.2 Borwein-Preiss variational principle

The Borwein-Preiss variational principle [44] is an important tool in infinite dimensional nonsmooth analysis. This basic result is strongly related with Stegall's variational principle [278], *smooth bumps* on Banach spaces, Smulyan's test describing the relationship between Fréchet differentiability and the strong extremum, properties of continuous convex functions on separable Asplund spaces, variational characterizations of Banach spaces, the Bishop-Phelps theorem or Phelps' lemma [239]. The generalized version we present here is due to Loewen and Wang [195] and enables us to deduce the standard form of the Borwein-Preiss variational principle, as well as other related results.

Let X be a Banach space and assume that $\rho : X \to [0, \infty)$ is a continuous function satisfying

$$\rho(0) = 0 \quad \text{and} \quad \rho_M := \sup\{\|x\|;\ \rho(x) < 1\} < +\infty\,. \tag{1.2}$$

An example of function with these properties is $\rho(x) = \|x\|^p$ with $p > 0$.

Given the families of real numbers $\mu_n \in (0, 1)$ and vectors $e_n \in X$ $(n \geq 0)$, we associate to ρ the *penalty function* ρ_∞ defined for all $x \in X$,

$$\rho_\infty(x) = \sum_{n=0}^{\infty} \rho_n(x - e_n), \quad \text{where } \rho_n(x) := \mu_n \rho((n+1)x)\,. \tag{1.3}$$

Definition 1.2 For the function $f : X \to \mathbb{R} \cup \{+\infty\}$, a point $x_0 \in X$ is a strong minimizer if $f(x_0) = \inf_X f$ and every minimizing sequence (z_n) of f satisfies $\|z_n - x_0\| \to 0$ as $n \to \infty$.

We observe that any strong minimizer of f is, in fact, a strict minimizer, that is $f(x) > f(x_0)$ for all $x \in X \setminus \{x_0\}$. The converse is true, as shown by $f(x) = x^2 e^x$, $x \in \mathbb{R}$, $x_0 = 0$.

The generalized version of the Borwein-Preiss variational principle due to Loewen and Wang is the following.

Theorem 1.5 *Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a lower semi-continuous function. Assume that $x_0 \in X$ and $\varepsilon > 0$ satisfy*

$$f(x_0) < \varepsilon + \inf_X f\,.$$

Let $(\mu_n)_{n \geq 0}$ be a decreasing sequence in $(0, 1)$ such that the series $\sum_{n=0}^{\infty} \mu_n$ is convergent. Then for any continuous function ρ satisfying (1.2), there exists a sequence $(e_n)_{n \geq 0}$ in X converging to e such that

(i) $\rho(x_0 - e) < 1$;
(ii) $f(e) + \varepsilon\rho_\infty(e) \le f(x_0)$;
(iii) *e is a strong minimizer of $f + \varepsilon\rho_\infty$. In particular, e is a strict minimizer of $f + \varepsilon\rho_\infty$, that is,*

$$f(e) + \varepsilon\rho_\infty(e) < f(x) + \varepsilon\rho_\infty(x) \qquad \text{for all } x \in X \setminus \{e\}.$$

Proof Define the sequence $(f_n)_{n\ge 0}$ such that $f_0 = f$ and for any $n \ge 0$,

$$f_{n+1}(x) := f_n(x) + \varepsilon\rho_n(x - e_n)\,.$$

Then $f_n \le f_{n+1}$ and f_n is lower semi-continuous.

Set $e_0 = x_0$. We observe that for any $n \ge 0$,

$$\inf_X f_{n+1} \le f_{n+1}(e_n) = f_n(e_n)\,. \tag{1.4}$$

If this inequality is strict, then there exists $e_{n+1} \in X$ such that

$$f_{n+1}(e_{n+1}) \le \frac{\mu_{n+1}}{2}\, f_n(e_n) + \left(1 - \frac{\mu_{n+1}}{2}\right) \inf_X f_{n+1} \le f_n(e_n)\,. \tag{1.5}$$

If equality holds in relation (1.4) then (1.5) also holds, but for e_{n+1} replaced with e_n. Consequently, there exists a sequence $(e_n)_{n\ge 0}$ in X such that relation (1.5) holds true.

Set

$$D_n := \left\{x \in X;\ f_{n+1}(x) \le f_{n+1}(e_{n+1}) + \frac{\varepsilon\mu_n}{2}\right\}\,.$$

Then D_n is not empty, since $e_{n+1} \in D_n$. By the lower semi-continuity of functions f_n we also deduce that D_n is a closed set. Since $\mu_{n+1} \in (0,1)$, relation (1.5) implies

$$\begin{aligned} f_{n+1}(e_{n+1}) - \inf_X f_{n+1} &\le \frac{\mu_{n+1}}{2}\left[f_n(e_n) - \inf_X f_{n+1}\right] \\ &\le f_n(e_n) - \inf_X f_n\,. \end{aligned} \tag{1.6}$$

We also observe that

$$f_0(e_0) - \inf_X f_0 = f(x_0) - \inf_X f < \varepsilon\,.$$

Next, we prove that

$$\text{the sequence } (D_n)_{n\ge 0} \text{ is decreasing} \tag{1.7}$$

and

$$\operatorname{diam}(D_n) \to 0 \quad \text{as } n \to \infty. \tag{1.8}$$

In order to prove (1.7), assume that $x \in D_n$, $n \geq 1$. Since the sequence $(\mu_n)_{n\geq 0}$ is decreasing, relation (1.5) implies

$$f_n(x) \leq f_{n+1}(x) \leq f_{n+1}(e_{n+1}) + \frac{\varepsilon\mu_n}{2} \leq f_n(e_n) + \frac{\varepsilon\mu_{n-1}}{2}\,,$$

hence $x \in D_{n-1}$.

Since $f_n \geq f_{n-1}$, relations (1.5) and (1.6) imply

$$\begin{aligned} f_n(e_n) - \inf_X f_n \quad &\leq \frac{\mu_n}{2}\left[f_{n-1}(e_{n-1}) - \inf_X f_n\right] \\ &\leq \frac{\mu_n}{2}\left[f_{n-1}(e_{n-1}) - \inf_X f_{n-1}\right] < \frac{\varepsilon\mu_n}{2}\,. \end{aligned} \tag{1.9}$$

For any $x \in D_n$, combining relation (1.9) and the definitions of f_{n+1} and D_n we obtain

$$\begin{aligned} \varepsilon\mu_n\rho((n+1)(x-e_n)) \quad &\leq f_{n+1}(e_{n+1}) - f_n(x) + \frac{\varepsilon\mu_n}{2} \\ &\leq f_{n+1}(e_{n+1}) - \inf_X f_n + \frac{\varepsilon\mu_n}{2} \\ &\leq f_n(e_n) - \inf_X f_n + \frac{\varepsilon\mu_n}{2} < \varepsilon\mu_n\,. \end{aligned} \tag{1.10}$$

Therefore $\rho((n+1)(x-e_n)) < 1$. So, by (1.2),

$$(n+1)\|x - e_n\| \leq \rho_M\,,$$

which shows that $\operatorname{diam}(D_n) \leq 2\rho_M/(n+1)$. This implies (1.8).

Since D_n is a closed set for any $n \geq 1$, then (1.7) and (1.8) imply that $\cap_{n=1}^{\infty} D_n$ contains a single point, denoted by e. Then $e_n \to e$ as $n \to \infty$. Thus, using $\rho((n+1)(x-e_n)) < 1$ for all $n \geq 0$ and $x \in X$, we deduce that $\rho(x_0 - e) < 1$.

Since the sequence $(f_n(e_n))_{n\geq 0}$ is nonincreasing and $f_0(e_0) = f(x_0)$ it follows that, in order to prove (ii), it is enough to deduce that

$$f(e) + \varepsilon\rho_\infty(e) \leq f_n(e_n)\,. \tag{1.11}$$

For this purpose we define the nonempty closed sets

$$C_n := \{x \in X;\ f_{n+1}(x) \leq f_{n+1}(e_{n+1})\}\,.$$

Since $f_n \leq f_{n+1}$ and $f_n(e_n) \geq f_{n+1}(e_{n+1})$ for all n, it follows that the sequence $(C_n)_{n\geq 0}$ is nested and $C_n \subset D_n$ for all n. Therefore $\cap_{n=0}^{\infty} C_n = \{e\}$ and

$$f_m(e) \leq f_m(e_m) \leq f_n(e_n) \leq f(x_0) \qquad \text{provided that } m > n. \tag{1.12}$$

Taking $m \to \infty$ we obtain (1.11).

It remains to argue that e is a strong minimizer of $f_\varepsilon := f + \varepsilon\rho_\infty$. Since

$$f_\varepsilon(x) \le \inf_X f_\varepsilon + \frac{\varepsilon\mu_n}{2}\,,$$

relation (1.12) yields

$$f_{n+1}(x) \le f_\varepsilon(x) \le f_\varepsilon(e) + \frac{\varepsilon\mu_n}{2} \le f_{n+1}(e_{n+1}) + \frac{\varepsilon\mu_n}{2}\,.$$

Setting

$$A_n := \left\{x \in X;\ f_\varepsilon(x) \le \frac{\varepsilon\mu_n}{2} + \inf_X f\right\}\,,$$

the above relation shows that $A_n \subset D_n$. So, by (1.8), we deduce that $\operatorname{diam}(A_n) \to 0$ as $n \to \infty$ which shows that e is a strong minimizer of $f_\varepsilon := f + \varepsilon\rho_\infty$. □

Assume that $p \ge 1$ and $\lambda > 0$. Taking

$$\rho(x) = \frac{\|x\|^p}{\lambda^p} \qquad \text{and} \qquad \mu_n = \frac{1}{2^{n+1}(n+1)}$$

we obtain the initial smooth version of the Borwein-Preiss variational principle. Roughly speaking, it asserts that the Lipschitz perturbations obtained in Ekeland's variational principle can be replaced by *superlinear perturbations* in a certain class of admissible functions.

Theorem 1.6 *Given $f : X \to \mathbb{R} \cup \{+\infty\}$ lower semi-continuous function, $x_0 \in X$, $\varepsilon > 0$, $\lambda > 0$, and $p \ge 1$, suppose*

$$f(x_0) < \varepsilon + \inf_X f\,.$$

Then there exists a sequence $(\mu_n)_{n\ge 0}$ with $\mu_n \ge 0$, $\sum_{n=0}^{\infty} \mu_n = 1$, and a point e in X, expressible as the limit of some sequence (e_n), such that for all $x \in X$,

$$f(x) + \frac{\varepsilon}{\lambda^p}\sum_{n=0}^{\infty} \mu_n \|x - e_n\|^p \ge f(e) + \frac{\varepsilon}{\lambda^p}\sum_{n=0}^{\infty} \mu_n \|e - e_n\|^p\,.$$

Moreover, $\|x_0 - e\| < \lambda$ and $f(e) \le \varepsilon + \inf_X f$.

We have seen in Corollary 1.2 of Ekeland's variational principle that any smooth bounded from below functional on a Banach space admits a sequence of "almost critical points". The next consequence of Borwein-Preiss' variational principle asserts that, in the framework of Hilbert spaces, such a functional admits a sequence of *stable* "almost critical points".