Dorian Goldfeld · Jay Jorgenson
Peter Jones · Dinakar Ramakrishnan
Kenneth A. Ribet · John Tate *Editors*

Number Theory, Analysis and Geometry

In Memory of Serge Lang

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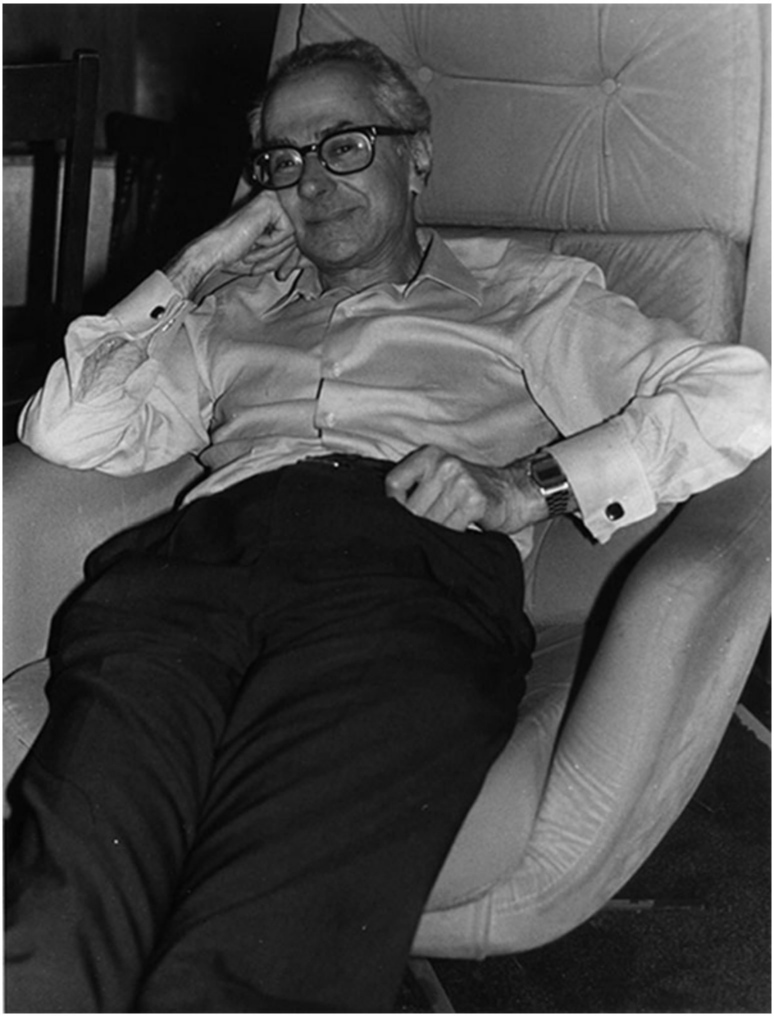
ISBN 978-1-4614-1259-5 e-ISBN 978-1-4614-1260-1
DOI 10.1007/978-1-4614-1260-1
Springer New York Dordrecht Heidelberg London

Library of Congress Control Number: 2011941121

Printed on acid-free paper

Springer is part of Springer Science+Business Media (www.springer.com)



Serge Lang (Photo provided courtesy of Kenneth A. Ribet.)

Preface

Serge Lang was an iconic figure in mathematics, both for his own important work and, perhaps even more crucially, for the indelible impact he left on the field, and on his students and colleagues. It would be difficult to find a mathematician who came of age in the past forty years, who had not been exposed to Serge's articles, monographs, and textbooks. Serge's writing shaped the mathematical perspectives of all who came in contact with them. Many were challenged by the glimpses of open problems and conjectures that Serge interweaved with his expositions of established subjects. Serge's exposition invariably transcended our discipline's preference for brevity and perfection, which often obscures the intuition underlying the subject. Serge was never one to conform.

One of Serge's uplifting qualities was his openness to new areas of mathematics and his concurrent willingness, even eagerness, to learn novel concepts and techniques. He was constantly reinventing himself, while sharing his accumulated wisdom with students and young mathematicians. Over the course of his career, he traversed a tremendous amount of mathematical ground. As he moved from subject to subject, he found analogies that led to important questions in such areas as number theory, arithmetic geometry, and the theory of negatively curved spaces. Lang's conjectures will keep many mathematicians occupied far into the future.

This memorial volume contains articles in a variety of areas of mathematics, attempting to represent Serge's breadth of interest and impact. We are happy to publish here (for the first time) Serge's final paper, *The heat kernel, theta inversion, and zetas on* $\Gamma\backslash G/K$, written jointly with one of us (J. Jorgenson). Except for that one article, which was left in the form it assumed just before Serge's passing, every other entry here was thoroughly refereed. We thank all the authors for their contributions to the volume and for their willingness to put up and comply with our demands for revision. Thanks also to the anonymous referees for their excellent and timely work.

We, the editors, are pleased to be a part of this production, especially since we were all fortunate enough to know Serge personally. We thank Stacey Croomes, the math administrator at Caltech, for her invaluable help in organizing the receipt of the articles, the refereeing process, and the revisions. We are grateful to Ann Kostant and Elizabeth Loew of Springer for their enthusiasm and helpful advice during the many months of editorial preparation. It took a village to produce this volume.

Columbia University Dorian Goldfeld
Yale University Peter Jones
The City College of New York Jay Jorgenson
Caltech Dinakar Ramakrishnan
University of California, Berkeley Kenneth A. Ribet
Harvard University John Tate

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Publications of Serge Lang: from 2000 and beyond

The five volumes of Serge Lang's *Collected Papers* from 1952 to 1999 were published by Springer-Verlag and noted below. His additional publications from 2000 can be found on the website of the American Mathematical Society's MathSciNet and are listed here.

Titles of books and journal articles are in italics.

[2000a] *Collected Papers.* Vol. I. 1952–1970. Springer-Verlag, New York, 2000. xxiv+525 pp.

[2000b] *Collected Papers.* Vol. II. 1971–1977. Springer-Verlag, New York, 2000. xvi+590 pp.

[2000c] *Collected Papers.* Vol. III. 1978–1990. Springer-Verlag, New York, 2000. xvi+393 pp.

[2000d] *Collected Papers.* Vol. IV. 1990–1996. Springer-Verlag, New York, 2000. xvi+471 pp.

[2001a] *Collected Papers.* Vol. V. 1993–1999. With Jay Jorgenson. Springer-Verlag, New York, 2001. xvi+426 pp.

[2001b] (with Jay Jorgenson). Guinand's theorem and functional equations for the Cramér functions. *J. Number Theory* **86** (2001), no. 2, 351–367.

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[2001c] (with Jay Jorgenson). The ubiquitous heat kernel. *Mathematics Unlimited—2001 and Beyond*, Springer, Berlin, 2001, pp. 655–683.

[2001d] Comments on non-references in Weil's works. *Gaz. Math.* No. 90 (2001), 46–52. 01A80.

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[2002b] *Algebra.* Revised third edition. Graduate Texts in Mathematics, Vol. 211. Springer-Verlag, New York, 2002. xvi+914 pp.

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[2002d] *Introduction to Differentiable Manifolds.* Second Edition. Universitext. Springer-Verlag, New York, 2002. xii+250 pp.

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[2003a] (with Jay Jorgenson). Analytic continuation and identities involving heat, Poisson, wave and Bessel kernels. *Math. Nachr.* **258** (2003), 44–70.

[2003b] (with Jay Jorgenson). *Spherical inversion on* $\mathrm{SL}_2(\mathbf{C})$. *Heat kernels and analysis on manifolds, graphs, and metric spaces* (Paris, 2002), 241–270, Contemp. Math., 338, Amer. Math. Soc., Providence, RI, 2003.

[2005a] (with Jay Jorgenson). A Gaussian space of test functions. *Math. Nachr.* **278** (2005), no. 7-8, 824–832.

[2005b] On the AMS Notices publication of Krieger's translation of Weil's 1940 letter [MR2125268]. *Notices Amer. Math. Soc.* 52 (2005), no. 6, 612–622.

[2005c] (with Jay Jorgenson). $\mathrm{Pos}_n(\mathrm{R})$ and Eisenstein series. Lecture Notes in Mathematics, 1868. Springer-Verlag, Berlin, 2005. viii+168 pp.

[2008] (with Jay Jorgenson). *The heat kernel and theta inversion on* $\mathrm{SL}_2(\mathbf{C})$. Springer Monographs in Mathematics. Springer, New York, 2008. x+319 pp.

[2009] (with Jay Jorgenson). *Heat Eisenstein series on* $\mathrm{SL}_n(\mathbf{C})$. Mem. Amer. Math. Soc. 201 (2009), no. 946, viii+127 pp.

[2010] (with Jay Jorgenson). *Heat kernel, theta inversions, and zetas on* $\Gamma\backslash G/K$. In this volume.

Additionally, the following articles about Lang appeared after September 2005.

[2006a] Marc Hindry: La géométrie diophantienne, selon Serge Lang. (French) [Diophantine geometry according to Serge Lang] *Gaz. Math.* No. 1**108** (2006), 17–32.

[2006b] David E. Rohrlich: Serge Lang. *Gaz. Math.* No. **108** (2006), 33–34.

[2006c] Michel Waldschmidt: Les contributions de Serge Lang à la thorie des nombres transcendants. (French) [Serge Lang's contributions to the theory of transcendental numbers] *Gaz. Math.* No. **108** (2006), 35–46.

[2006d] Jay Jorgenson and Steven G. Krantz: Serge Lang, 1927–2005. *Notices Amer. Math. Soc.* 53 (2006), no. 5, 536–553.

[2006e] Obituary: Serge Lang (1927–2005). (Spanish) *Lect. Mat.* **27** (2006), no. 2, 166–167.

[2007] Jay Jorgenson and Steven G. Krantz: The mathematical contributions of Serge Lang. *Notices Amer. Math. Soc.* 54 (2007), no. 4, 476–497.

Introduction

John Tate

This introduction is meant as a brief account of Serge Lang's life and his enormously varied contributions to mathematics. Much more about this remarkable man can be found in two articles in the *Notices of the AMS* by Jay Jorgenson and Steven G. Krantz. The first of these, "Serge Lang, 1927–2005" (May 2006) contains a fuller account of Lang's life than we can give here, and includes memories of Serge by twenty-two of his friends, students, colleagues, and even some whom Serge might have seen as adversaries. Read together, these short pieces give a vivid image of Lang. The second article, "Mathematical Contributions of Serge Lang" (April 2007), contains an overview of his research, followed by discussions of its different aspects by seven colleagues in the various fields. These articles, and conversations with Lang's friends Dick Gross, Dinakar Ramakrishnan, and Ken Ribet, have been of great help to me in writing this introduction.

Lang spent his childhood in Saint-Germain-en-Laye, a western suburb of Paris famous for its chateau and long terrace with a view over the valley of the Seine and Paris in the distance. Lang's teen years were spent in quite different surroundings. After emigrating with his family to Los Angeles, he attended Beverly Hills High and Caltech, graduating in 1946 with a BA in physics.

He then did a year and a half of military service with the U.S. Army in Europe. This was of great help to him in his future career, for he served in a clerical position in which he learned to type at incredible speed.

Next, Serge enrolled in the graduate philosophy program at Princeton. Fortunately for mathematics he was disappointed by the quality of the philosophy seminars, and managed to switch to the math graduate program the following year. I don't know why he chose mathematics, but the Princeton math program was an outstanding one and student morale was very high.

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Though he was not particularly well prepared, Serge plunged right in. For example, knowing little number theory, he attended Emil Artin's seminar on class field theory during his first year and was fascinated. Math, especially algebra and number theory, were the subjects for him! He soon became one of Artin's Ph.D. students along with me and a few others that Artin had brought with him from Indiana. We all felt very fortunate to have Artin as our advisor. In the foreword to his collected works, Lang writes, "I take this opportunity to express once more my appreciation for having been Artin's student. I could not have had a better start in my mathematical life."

Lang got his Ph.D in 1951 with a thesis on quasi-algebraic closure in which he proved that a field complete in a discrete valuation with algebraically closed residue field is quasi-algebraically closed, and that the same is true for several kinds of dense subfields of such a field. Another result in his thesis is a key ingredient in the proof of the Ax–Kochen theorem,[1] which can be viewed as a corrected version of Artin's conjecture that p-adic fields have property C_2. For each degree d, this is true for all but a finite set of primes p, but not necessarily for all, as Artin had guessed.

Lang stayed in Princeton for two more years, with postdoc positions at the university and at the Institute for Advanced Study. Then, after two years of an instructorship at the University of Chicago, where he interacted with Weil and his circle, he accepted a permanent position at Columbia University. He stayed there for the next fifteen years except for a Fulbright Fellowship year in Paris during 1957–58. In 1970, Lang resigned his position at Columbia in protest of the university's handling of student unrest during 1968–69. After visiting professorships at Princeton and Harvard, he accepted a permanent position at Yale. He retired from Yale in 2005, a few months before his death. That is a bare-bones account of Lang's life and the way he got into mathematics.

Serge Lang contributed to mathematics in so many ways that it's hard to know where to begin. Let's start with some remarks on his research, with no attempt to cover it completely. He published his *Collected Papers* in five volumes with Springer in 2000. They contain all of his research papers through 1999, together with reprints of a few of his Springer Lecture Notes and two of his books that were out of print. There are also some interesting accounts of some special topics on which Lang held strong views, especially in Volume IV.

In a brief foreword in Volume 1, Lang gives his own classification of his work into periods as follows:

1. 1951–1954 Thesis on quasi-algebraic closure and related matters.
2. 1954–1962 Algebraic geometry and abelian (or group) varieties; geometric class field theory.[2]

[1] As Deligne pointed out, I misstated the Ax–Kochen theorem in my *Notices* article on Lang's early work, by interchanging "prime p" and "degree d." I hope this "senior moment" misled no one.

[2] This work was the beginning of higher-dimensional class field theory and earned Lang a Cole Prize in 1959.

3. 1963–1975	Transcendental numbers and Diophantine approximation on algebraic groups.
4. 1970	First paper on analytic number theory—jump to Jorgenson–Lang.
5. 1975	$SL_2(\mathbf{R})$—jump to Jorgenson–Lang.
6. 1972–1977	Lang–Trotter Frobenius distributions.
7. 1973–1981	Modular curves, Kubert–Lang modular units.
8. 1974, 1982–1991	Diophantine geometry, complex hyperbolic spaces, and Nevanlinna theory.
9. 1985, 1988	Riemann–Roch and Arakelov theory.
10. 1992–2000+	Jorgenson–Lang (analytic number theory and connections with spectral analysis, heat kernel, differential geometry, Lie groups, and symmetric spaces).

The arrangement of the 2007 *Notices* article fits quite well with Lang's scheme. Here is a list of the authors in the order of their appearance, followed by the periods of Lang's work they discuss. Tate 1,2; Buium 1,2,3; Waldschmidt 3; Rohrlich 6,7; Vojta 8,9; Jorgenson 10; and Kim, who wrote on the theme of the fundamental group in Lang's work, rather than on a specific period.

Lang wrote over 70 research papers and proved many important theorems, but of at least equal significance were his conjectures, his points of view, and his way of looking at things. Waldschmidt expresses this well (loc. cit.), writing about Lang's work on transcendental numbers:

> With his outstanding insight and his remarkable pedagogical gifts, Lang comes into the picture and contributes to the subject in at least two very different ways: on the one hand, he simplifies the arguments (sometimes excessively) and produces the first very clear proofs which can be taught easily; on the other hand, he introduces new tools, like group varieties, which put the topic closer to the interests of many a mathematician.

Waldschmidt concludes his article as follows:

> Among the contributions of Lang to transcendental number theory (also to Diophantine geometry), the least are not his many conjectures which shed a new light on the subject. On the contrary, he had a way of considering what the situation should be, which was impressive. Indeed, he succeeded in getting rid of the limits from the existing results and methods. He made very few errors in his predictions, especially if we compare them with the large number of conjectures he proposed. His description of the subject will be a guideline for a very long time.

As Vojta points out, the title of Lang's magnum opus, *Fundamentals of Diophantine Geometry* suggests that Serge's outlook on number theory was decidedly geometric. Mazur puts this beautifully in concluding his memory of Serge article in the *Notices*:

> Over the decades of mathematics, Lang was led, more specifically, by an over-arching vision, which he pursued through the agency of various fields of mathematics. The vision, baldly put, is that *geometry* is an extraordinarily striking dictator of qualitative *dipohantine* behavior. The still open *Conjecture of Lang* in higher dimensions continues to serve as a

> guiding principle to the way in which the grand subjects of geometry and number theory meet, just as Serge himself served as an inspiror of generations of mathematicians, and a spokesman for intellectual honesty.

The conjecture of Lang to which Mazur refers is easy to state. A projective algebraic variety V defined over a number field $F \subset \mathbf{C}$ is *Mordellic* if and only if the corresponding complex space $V(\mathbf{C})$ is *hyperbolic*. Here Mordellic means that for each finite extension field E of F, the set $V(E)$ of points of V with coordinates in E is finite. Hyperbolic meant for Lang, when he made the conjecture in 1974, that the Kobayashi semidistance on $V(\mathbf{C})$ is actually a distance, but we now know, thanks to Brody (1978), that this property is equivalent to there being no nonconstant holomorphic map $\mathbf{C} \to V(\mathbf{C})$. A Riemann surface is hyperbolic if and only if its genus is ≥ 2, so that for curves, Lang's conjecture is equivalent to the famous Mordell conjecture proved by Faltings in 1983. In higher dimensions the conjecture is still open, though it has been proved for closed subvarieties of abelian varieties.

In the 1980s Lang thought deeply about the Mordellic–hyperbolic relationship and introduced plausible variants of the above conjecture which have turned out to have very interesting unexpected implications, such as the existence of a bound $B(g, F)$ depending only on g and F for the number of rational points on a curve of genus $g \geq 2$ defined over a number field F.

Serge led a regular life. During the winter holidays he visited his sister in Los Angeles. He spent the early summer in Europe and July–August in Berkeley, where he had an apartment. In Europe he spent a month in one place, Paris in the early years, Bonn later in his life, but also visited regularly other mathematical centers, Zurich, Berlin, Moscow.... In Berkeley he interacted with the large community of resident and visiting mathematicians.

Lang was an effective communicator, an excellent source of mathematical news. Dick Gross likens his gathering and distributing information to the cross-pollination of a bumblebee. Serge kept in touch with friends in many places, not only in person, by traveling, but by phone. If he had a question or thought for anyone anywhere in the world, he just picked up the phone. When Yale was considering an offer to Serge, I remember warning the department that if he accepted, its phone bill would at least double, but that in fact, the phoning he would be doing was just one more reason for making the offer.

A more important reason for Yale's doing so is that Lang was an excellent and caring teacher. This was recognized by his being awarded the Dylon Hixon Prize for Teaching Excellence in the Natural Sciences at Yale. We were reminded of the esteem in which his former students held him by the testimony of so many of them at the memorial meeting for Lang at Yale in February 2006. One of them, Anthony Petrello, announced the establishment of a Yale fund for an annual prize in Lang's honor which he was launching with a large seed contribution, and the promise of matching funds.

Another memorial to Lang is the Serge Lang Undergraduate Lecture Series at Berkeley. There, when students returned to classes at the end of August, Serge often gave talks to the Math Undergraduate Student Association (MUSA). His talks over the years were incorporated into his 1999 volume "Math talks for Undergraduates." These talks became formalized, and each year from 1999 to 2005 Serge gave a MUSA lecture at 4pm on the first day of classes. The lecture on August 25, 2005, "Weierstrass–Dirac Families," shortly before his death, was one of Serge Lang's last mathematics talks. In response to MUSA's request to somehow continue this tradition, the Department of Mathematics inaugurated the Serge Lang Undergraduate Lecture Series, each fall inviting someone to give a lecture for undergraduates. Ken Ribet made this happen and Anthony Petrello contributed a major part of the initial funding.

Serge was known not only for his support of his students, but also of his younger colleagues at the start of their careers. John Coates thought one of Serge's most remarkable qualities was his unstinting support of young mathematicians. Barry Mazur, after recounting his first encounter with Serge, writes:

> And Serge did this sort of thing through the decades, with many of the young; he would proffer to them gracious, yet demanding, invitations to engage as a genuine colleague—not teacher to student, but mathematician to mathematician; he did all this naturally, and with extraordinary generosity and success.

Lang was awarded the 1999 AMS Leroy P. Steele Prize for Mathematical Exposition "for his many mathematics books." The amount of mathematical knowledge that has been made accessible to students of all ages all over the world by Lang's more than 40 books is amazing to contemplate. Their range both in subject and in level is astonishingly broad. Most were new and modern for their time with Lang's insistence on functoriality and axiomatization. He was almost unique in the way he regularly learned new topics throughout his life, topics often not close to his main interests (algebra and number theory), and wrote textbooks on them (such as "Differentiable Manifolds" and $SL_2(\mathbf{R})$), thereby influencing new generations of students pursuing those fields. He brought excitement to his books that challenged readers to rise above themselves by tackling them.

Lang's Algebra is a classic, still the best reference book in algebra in print today. The first edition in 1965 has been kept up-to-date with new editions and revisions. The latest is the corrected fourth printing of the revised third edition (2004). For Lang, the important thing in a book is its timeliness, and its global aspects, such as arrangement of topics, and degree of abstraction. He did not worry much about an occasional error in a proof, and was widely criticized for this. Given the short time he spent writing a book, there are relatively few of these oversights, and when he became aware of one he was highly attentive to its correction in the next edition or the next printing. These oversights could be useful. I remember a few times first recognizing a student who turned out to be very strong when s/he came for help in understanding one of Lang's faulty arguments.

Lang was driven to publish. In addition to his own writing, he saw to the publication of at least two books which were not his own, namely *Class Field Theory* by Artin and me, and the Collected Papers of Emil Artin. He should really have been included among the authors of the former, for the main part of it is essentially Serge's rewriting of his own notes from the 1951–52 Princeton seminar, and the