UNIVERSITY OF ABERDEEN

Department of Mathematical Sciences

Advanced Calculus and Analysis MA1002

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Foreword

These Notes

The notes contain the material that I use when preparing lectures for a course I gave from the mid 1980's until 1994; in that sense they are *my* lecture notes.

> ”Lectures were once useful, but now when all can read, and books are so numerous, lectures are unnecessary.” *Samuel Johnson, 1799.*

Lecture notes have been around for centuries, either informally, as handwritten notes, or formally as textbooks. Recently improvements in typesetting have made it easier to produce “personalised” printed notes as here, but there has been no fundamental change. Experience shows that very few people are able to use lecture notes as a *substitute* for lectures; if it were otherwise, lecturing, as a profession would have died out by now.

These notes have a long history; a “first course in analysis” rather like this has been given within the Mathematics Department for at least 30 years. During that time many people have taught the course and all have left their mark on it; clarifying points that have proved difficult, selecting the “right” examples and so on. I certainly benefited from the notes that Dr Stuart Dagger had written, when I took over the course from him and this version builds on that foundation, itslef heavily influenced by (Spivak 1967) which was the recommended textbook for most of the time these notes were used.

The notes are written in LaTeX which allows a higher level view of the text, and simplifies the preparation of such things as the index on page 101 and numbered equations. You will find that most equations are not numbered, or are numbered symbolically. However sometimes I want to refer back to an equation, and in that case it is numbered within the section. Thus Equation (1.1) refers to the first numbered equation in Chapter 1 and so on.

Acknowledgements

These notes, in their printed form, have been seen by many students in Aberdeen since they were first written. I thank those (now) anonymous students who helped to improve their quality by pointing out stupidities, repetitions misprints and so on.

Since the notes have gone on the web, others, mainly in the USA, have contributed to this gradual improvement by taking the trouble to let me know of difficulties, either in content or presentation. As a way of thanking those who provided such corrections, I endeavour to incorporate the corrections in the text almost immediately. At one point this was no longer possible; the diagrams had been done in a program that had been ‘subsequently “upgraded” so much that they were no longer useable. For this reason I had to withdraw the notes. However all the diagrams have now been redrawn in “public

domaian" tools, usually `xfig` and `gnuplot`. I thus expect to be able to maintain them in future, and would again welcome corrections.

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Chapter 1

Introduction.

This chapter contains reference material which you should have met before. It is here both to remind you that you have, and to collect it in one place, so you can easily look back and check things when you are in doubt.

You are aware by now of just how sequential a subject mathematics is. If you don't understand something when you first meet it, you usually get a second chance. Indeed you will find there are a number of ideas here which it is essential you now understand, because you will be using them all the time. So another aim of this chapter is to repeat the ideas. It makes for a boring chapter, and perhaps should have been headed "all the things you hoped never to see again". However I am only emphasising things that you will be using in context later on.

If there is material here with which you are not familiar, don't panic; any of the books mentioned in the book list can give you more information, and the first tutorial sheet is designed to give you practice. And ask in tutorial if you don't understand something here.

1.1 The Need for Good Foundations

It is clear that the calculus has many outstanding successes, and there is no real discussion about its viability as a theory. However, despite this, there are problems if the theory is accepted uncritically, because naive arguments can quickly lead to errors. For example the chain rule can be phrased as

$$\frac{df}{dx} = \frac{df}{dy}\frac{dy}{dx},$$

and the "quick" form of the proof of the chain rule — cancel the dy's — seems helpful. However if we consider the following result, in which the pressure P, volume V and temperature T of an enclosed gas are related, we have

$$\frac{\partial P}{\partial V}\frac{\partial V}{\partial T}\frac{\partial T}{\partial P} = -1, \tag{1.1}$$

a result which certainly does not appear "obvious", even though it is in fact true, and we shall prove it towards the end of the course.

Another example comes when we deal with infinite series. We shall see later on that the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} \dots$$

adds up to $\log 2$. However, an apparently simple re-arrangement gives

$$\left(1 - \frac{1}{2}\right) - \frac{1}{4} + \left(\frac{1}{3} - \frac{1}{6}\right) - \frac{1}{8} + \left(\frac{1}{5} - \frac{1}{10}\right) \dots$$

and this clearly adds up to half of the previous sum — or $\log(2)/2$.

It is this need for care, to ensure we can *rely* on calculations we do, that motivates much of this course, illustrates why we emphasise accurate argument as well as getting the "correct" answers, and explains why in the rest of this section we need to revise elementary notions.

1.2 The Real Numbers

We have four infinite sets of familiar objects, in increasing order of complication:

$\mathbb{N}$ — **the Natural numbers** are defined as the set $\{0, 1, 2, \dots, n, \dots\}$. Contrast these with the **positive integers**; the same set without 0.

$\mathbb{Z}$ — **the Integers** are defined as the set $\{0, \pm1, \pm2, \dots, \pm n, \dots\}$.

$\mathbb{Q}$ — **the Rational numbers** are defined as the set $\{p/q : p, q \in \mathbb{Z}, q \neq 0\}$.

$\mathbb{R}$ — **the Reals** are defined in a much more complicated way. In this course you will start to see why this complication is necessary, as you use the distinction between $\mathbb{R}$ and $\mathbb{Q}$.

Note: We have a natural inclusion $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$, and each inclusion is proper. The only inclusion in any doubt is the last one; recall that $\sqrt{2} \in \mathbb{R} \setminus \mathbb{Q}$ (i.e. it is a real number that is not rational).

One point of this course is to illustrate the difference between $\mathbb{Q}$ and $\mathbb{R}$. It is subtle: for example when computing, it can be ignored, because a computer always works with a rational approximation to any number, and as such can't distinguish between the two sets. We hope to show that the complication of introducing the "extra" reals such as $\sqrt{2}$ is worthwhile because it gives simpler results.

Properties of $\mathbb{R}$

We summarise the properties of $\mathbb{R}$ that we work with.

Addition: We can add and subtract real numbers exactly as we expect, and the usual rules of arithmetic hold — such results as $x + y = y + x$.

Multiplication: In the same way, multiplication and division behave as we expect, and interact with addition and subtraction in the usual way. So we have rules such as $a(b+c) = ab + ac$. Note that we can divide by any number except 0. We make no attempt to make sense of $a/0$, even in the "funny" case when $a = 0$, so for us $0/0$ is meaningless. Formally these two properties say that (algebraically) $\mathbb{R}$ is a field, although it is not essential at this stage to know the terminology.

Order As well as the algebraic properties, $\mathbb{R}$ has an ordering on it, usually written as "$a > 0$" or "$\geq$". There are three parts to the property:

- **Trichotomy** For any $a \in \mathbb{R}$, exactly one of $a > 0$, $a = 0$ or $a < 0$ holds, where we write $a < 0$ instead of the formally correct $0 > a$; in words, we are simply saying that a number is either positive, negative or zero.
- **Addition** The order behaves as expected with respect to addition: if $a > 0$ and $b > 0$ then $a + b > 0$; i.e. the sum of positives is positive.
- **Multiplication** The order behaves as expected with respect to multiplication: if $a > 0$ and $b > 0$ then $ab > 0$; i.e. the product of positives is positive.

Note that we write $a \geq 0$ if either $a > 0$ or $a = 0$. More generally, we write $a > b$ whenever $a - b > 0$.

Completion The set $\mathbb{R}$ has an additional property, which in contrast is much more mysterious — it is complete. It is this property that distinguishes it from $\mathbb{Q}$. Its effect is that there are always "enough" numbers to do what we want. Thus there are enough to solve any algebraic equation, even those like $x^2 = 2$ which can't be solved in $\mathbb{Q}$. In fact there are (uncountably many) more - all the numbers like π, certainly not rational, but in fact not even an algebraic number, are also in $\mathbb{R}$. We explore this property during the course.

One reason for looking carefully at the properties of $\mathbb{R}$ is to note possible errors in manipulation. One aim of the course is to emphasise accurate explanation. Normal algebraic manipulations can be done without comment, but two cases arise when more care is needed:

> *Never divide by a number without checking first that it is non-zero.*

Of course we know that 2 is non zero, so you don't need to justify dividing by 2, but if you divide by x, you should always say, at least the first time, why $x \neq 0$. If you don't know whether $x = 0$ or not, the rest of your argument may need to be split into the two cases when $x = 0$ and $x \neq 0$.

> *Never multiply an inequality by a number without checking first that the number is positive.*

Here it is even possible to make the mistake with numbers; although it is perfectly sensible to multiply an equality by a constant, the same is not true of an inequality. If $x > y$, then of course $2x > 2y$. However, we have $(-2)x < (-2)y$. If multiplying by an expression, then again it may be necessary to consider different cases separately.

1.1. Example. Show that if $a > 0$ then $-a < 0$; and if $a < 0$ then $-a > 0$.

Solution. This is not very interesting, but is here to show how to use the properties formally.

Assume the result is false; then by trichotomy, $-a = 0$ (which is false because we know $a > 0$), or $(-a) > 0$. If this latter holds, then $a + (-a)$ is the sum of two positives and so is positive. But $a + (-a) = 0$, and by trichotomy $0 > 0$ is false. So the only consistent possibility is that $-a < 0$. The other part is essentially the same argument.

1.2. Example. Show that if $a > b$ and $c < 0$, then $ac < bc$.

Solution. This also isn't very interesting; and is here to remind you that the order in which questions are asked can be helpful. The hard bit about doing this is in Example 1.1. This is an idea you will find a lot in example sheets, where the next question *uses* the result of the previous one. It may dissuade you from dipping into a sheet; try rather to work through systematically.

Applying Example 1.1 in the case $a = -c$, we see that $-c > 0$ and $a - b > 0$. Thus using the multiplication rule, we have $(a - b)(-c) > 0$, and so $bc - ac > 0$ or $bc > ac$ as required.

1.3. Exercise. Show that if $a < 0$ and $b < 0$, then $ab > 0$.

1.3 Inequalities

One aim of this course is to get a *useful* understanding of the behaviour of systems. Think of it as trying to see the wood, when our detailed calculations tell us about individual trees. For example, we may want to know *roughly* how a function behaves; can we perhaps ignore a term because it is small and simplify things? In order to to this we need to estimate — replace the term by something bigger which is easier to handle, and so we have to deal with inequalities. It often turns out that we can "give something away" and still get a useful result, whereas calculating directly can prove either impossible, or at best unhelpful. We have just looked at the rules for manipulating the order relation. This section is probably all revision; it is here to emphasise the need for care.

1.4. Example. Find $\{x \in \mathbb{R} : (x - 2)(x + 3) > 0\}$.

Solution. Suppose $(x - 2)(x + 3) > 0$. Note that if the product of two numbers is positive then either both are positive or both are negative. So *either* $x - 2 > 0$ and $x + 3 > 0$, in which case both $x > 2$ and $x > -3$, so $x > 2$; *or* $x - 2 < 0$ and $x + 3 < 0$, in which case both $x < 2$ and $x < -3$, so $x < -3$. Thus

$$\{x : (x - 2)(x + 3) > 0\} = \{x : x > 2\} \cup \{x : x < -3\}.$$

1.5. Exercise. Find $\{x \in \mathbb{R} : x^2 - x - 2 < 0\}$.

Even at this simple level, we can produce some interesting results.

1.6. Proposition (Arithmetic - Geometric mean inequality). *If $a \geq 0$ and $b \geq 0$ then*

$$\frac{a + b}{2} \geq \sqrt{ab}.$$

Solution. For any value of x, we have $x^2 \geq 0$ (why?), so $(a-b)^2 \geq 0$. Thus

$$\begin{aligned} a^2 - 2ab + b^2 &\geq 0, \\ a^2 + 2ab + b^2 &\geq 4ab. \\ (a+b)^2 &\geq 4ab. \end{aligned}$$

Since $a \geq 0$ and $b \geq 0$, taking square roots, we have $\dfrac{a+b}{2} \geq \sqrt{ab}$. This is the **arithmetic - geometric mean inequality**. We study further work with inequalities in section 1.7.

1.4 Intervals

We need to be able to talk easily about certain subsets of $\mathbb{R}$. We say that $I \subset \mathbb{R}$ is an **open interval** if

$$I = (a, b) = \{x \in \mathbb{R} : a < x < b\}.$$

Thus an open interval excludes its end points, but contains all the points in between. In contrast a **closed interval** contains both its end points, and is of the form

$$I = [a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}.$$

It is also sometimes useful to have **half - open** intervals like $(a, b]$ and $[a, b)$. It is trivial that $[a, b] = (a, b) \cup \{a\} \cup \{b\}$.

> The two end points a and b are points in $\mathbb{R}$. It is *sometimes* convenient to allow also the possibility $a = -\infty$ and $b = +\infty$; it should be clear from the context whether this is being allowed. If these extensions are being excluded, the interval is sometimes called a *finite* interval, just for emphasis.

Of course we can easily get to more general subsets of $\mathbb{R}$. So $(1, 2) \cup [2, 3] = (1, 3]$ shows that the union of two intervals may be an interval, while the example $(1, 2) \cup (3, 4)$ shows that the union of two intervals *need not* be an interval.

1.7. Exercise. Write down a pair of intervals I_1 and I_2 such that $1 \in I_1$, $2 \in I_2$ and $I_1 \cap I_2 = \emptyset$.

Can you still do this, if you require in addition that I_1 is centred on 1, I_2 is centred on 2 and that I_1 and I_2 have the same (positive) length? What happens if you replace 1 and 2 by any two numbers l and m with $l \neq m$?

1.8. Exercise. Write down an interval I with $2 \in I$ such that $1 \notin I$ and $3 \notin I$. Can you find the largest such interval? Is there a largest such interval if you also require that I is closed?

Given l and m with $l \neq m$, show there is always an interval I with $l \in I$ and $m \notin I$.

1.5 Functions

Recall that $f : D \subset \mathbb{R} \to T$ is a **function** if $f(x)$ is a well defined value in T for each $x \in D$. We say that D is the **domain** of the function, T is the **target space** and $f(D) = \{f(x) : x \in D\}$ is the **range** of f.

Note first that the definition says nothing about a formula; just that the result must be properly defined. And the definition can be complicated; for example

$$f(x) = \begin{cases} 0 & \text{if } x \leq a \text{ or } x \geq b; \\ 1 & \text{if } a < x < b. \end{cases}$$

defines a function on the whole of $\mathbb{R}$, which has the value 1 on the open interval (a, b), and is zero elsewhere [and is usually called the *characteristic* function of the interval (a, b).]

In the simplest examples, like $f(x) = x^2$ the domain of f is the whole of $\mathbb{R}$, but even for relatively simple cases, such as $f(x) = \sqrt{x}$, we need to restrict to a smaller domain, in this case the domain D is $\{x : x \geq 0\}$, since we cannot define the square root of a negative number, at least if we want the function to have real - values, so that $T \subset \mathbb{R}$.

Note that the domain is part of the definition of a function, so changing the domain technically gives a different function. This distinction will start to be important in this course. So $f_1 : \mathbb{R} \to \mathbb{R}$ defined by $f_1(x) = x^2$ and $f_2 : [-2, 2] \to \mathbb{R}$ defined by $f_2(x) = x^2$ are formally *different* functions, even though they both are "x^2" Note also that the range of f_2 is $[0, 4]$. This illustrate our first use of intervals. Given $f : \mathbb{R} \to \mathbb{R}$, we can always restrict the domain of f to an interval I to get a new function. Mostly this is trivial, but sometimes it is useful.

Another natural situation in which we need to be careful of the domain of a function occurs when taking quotients, to avoid dividing by zero. Thus the function

$$f(x) = \frac{1}{x - 3} \quad \text{has domain } \{x \in \mathbb{R} : x \neq 3\}.$$

The point we have excluded, in the above case 3 is sometimes called a **singularity** of f.

1.9. Exercise. Write down the natural domain of definition of each of the functions:

$$f(x) = \frac{x - 2}{x^2 - 5x + 6} \qquad g(x) = \frac{1}{\sin x}.$$

Where do these functions have singularities?

It is often of interest to investigate the behaviour of a function near a singularity. For example if

$$f(x) = \frac{x - a}{x^2 - a^2} = \frac{x - a}{(x - a)(x + a)} \quad \text{for } x \neq a.$$

then since $x \neq a$ we can cancel to get $f(x) = (x + a)^{-1}$. This is of course a different representation of the function, and provides an indication as to how f may be extended through the singularity at a — by giving it the value $(2a)^{-1}$.

1.6 Neighbourhoods

This situation often occurs. We need to be able to talk about a function *near* a point: in the above example, we don't want to worry about the singularity at $x = -a$ when we are discussing the one at $x = a$ (which is actually much better behaved). If we only look at the points distant less than d for a, we are really looking at an interval $(a - d, a + d)$; we call such an interval a **neighbourhood** of a. For traditional reasons, we usually replace the distance d by its Greek equivalent, and speak of a distance δ. If $\delta > 0$ we call the interval $(a-\delta, a+\delta)$ a neighbourhood (sometimes a δ - neighbourhood) of a. The significance of a neighbourhood is that it is an interval in which we can look at the behaviour of a function without being distracted by other irrelevant behaviours. It usually doesn't matter whether δ is very big or not. To see this, consider:

1.10. Exercise. Show that an open interval contains a neighbourhood of each of its points.

We can rephrase the result of Ex 1.7 in this language; given $l \neq m$ there is some (sufficiently small) δ such that we can find disjoint δ - neighbourhoods of l and m. We use this result in Prop 2.6.

1.7 Absolute Value

Here is an example where it is natural to use a two part definition of a function. We write

$$|x| = \begin{cases} x & \text{if } x \geq 0; \\ -x & \text{if } x < 0. \end{cases}$$

An equivalent definition is $|x| = \sqrt{x^2}$. This is the **absolute value** or **modulus** of x. It's particular use is in describing distances; we interpret $|x-y|$ as the distance between x and y. Thus

$$(a-\delta, a+\delta) = \{X \in \mathbb{R} : |x-a| < \delta\},$$

so a δ - neighbourhood of a consists of all points which are closer to a than δ.

Note that we can always "expand out" the inequality using this idea. So if $|x-y| < k$, we can rewrite this without a modulus sign as the pair of inequalities $-k < x-y < k$. We sometimes call this "unwrapping" the modulus; conversely, in order to establish an inequality involving the modulus, it is simply necessary to show the corresponding pair of inequalities.

1.11. Proposition (The Triangle Inequality.). *For any $x, y \in \mathbb{R}$,*

$$|x+y| \leq |x| + |y|.$$

Proof. Since $-|x| \leq x \leq |x|$, and the same holds for y, combining these we have

$$-|x| - |y| \leq x+y \leq |x| + |y|$$

and this is the same as the required result. □

1.12. Exercise. Show that for any $x, y, z \in \mathbb{R}$, $|x-z| \leq |x-y| + |y-z|$.

1.13. Proposition. *For any $x, y \in \mathbb{R}$,*

$$|x-y| \geq \Big| |x| - |y| \Big|.$$

Proof. Using 1.12 we have

$$|x| = |x - y + y| \leq |x - y| + |y|$$

and so $|x| - |y| \leq |x - y|$. Interchanging the rôles of x and y, and noting that $|x| = |-x|$, gives $|y| - |x| \leq |x - y|$. Multiplying this inequality by -1 and combining these we have

$$-|x - y| \leq |x| - |y| \leq |x - y|$$

and this is the required result. □

1.14. Example. Describe $\{x \in \mathbb{R} : |5x - 3| > 4\}$.

Proof. Unwrapping the modulus, we have either $5x - 3 < -4$, or $5x - 3 > 4$. From one inequality we get $5x < -4 + 3$, or $x < -1/5$; the other inequality gives $5x > 4 + 3$, or $x > 7/5$. Thus

$$\{x \in \mathbb{R} : |5x - 3| > 4\} = (-\infty, -1/5) \cup (7/5, \infty).$$

□

1.15. Exercise. Describe $\{x \in \mathbb{R} : |x + 3| < 1\}$.

1.16. Exercise. Describe the set $\{x \in \mathbb{R} : 1 \leq x \leq 3\}$ using the absolute value function.

1.8 The Binomial Theorem and other Algebra

At its simplest, the binomial theorem gives an expansion of $(1+x)^n$ for any positive integer n. We have

$$(1 + x)^n = 1 + nx + \frac{n.(n-1)}{1.2}x^2 + \ldots + \frac{n.(n-1).(n-k+1)}{1.2.\ldots.k}x^k + \ldots + x^n.$$

Recall in particular a few simple cases:

$$\begin{aligned}(1 + x)^3 &= 1 + 3x + 3x^2 + x^3,\\ (1 + x)^4 &= 1 + 4x + 6x^2 + 4x^3 + x^4,\\ (1 + x)^5 &= 1 + 5x + 10x^2 + 10x^3 + 5x^4 + x^5.\end{aligned}$$

There is a more general form:

$$(a + b)^n = a^n + na^{n-1}b + \frac{n.(n-1)}{1.2}a^{n-2}b^2 + \ldots + \frac{n.(n-1).(n-k+1)}{1.2.\ldots.k}a^{n-k}b^k + \ldots + b^n,$$

with corresponding special cases. Formally this result is only valid for any positive integer n; in fact it holds appropriately for more general exponents as we shall see in Chapter 7

Another simple algebraic formula that can be useful concerns powers of differences:

$$\begin{aligned}a^2 - b^2 &= (a - b)(a + b),\\ a^3 - b^3 &= (a - b)(a^2 + ab + b^2),\\ a^4 - b^4 &= (a - b)(a^3 + a^2b + ab^2 + b^3)\end{aligned}$$

and in general, we have

$$a^n - b^n = (a-b)(a^{n-1} + a^{n-2}b + a^{n-3}b^2 + \ldots + a^b n - 1 + b^{n-1}).$$

Note that we made use of this result when discussing the function after Ex 1.9.

And of course you remember the usual "completing the square" trick:

$$\begin{aligned} ax^2 + bx + c &= a\left(x^2 + \frac{b}{a}x + \frac{b^2}{4a^2}\right) + c - \frac{b^2}{4a} \\ &= a\left(x + \frac{b}{2a}\right)^2 + \left(c - \frac{b^2}{4a}\right). \end{aligned}$$

Chapter 2

Sequences

2.1 Definition and Examples

2.1. Definition. A (real infinite) sequence is a map $a : \mathbb{N} \to \mathbb{R}$

Of course if is more usual to call a function f rather than a; and in fact we usually start labeling a sequence from 1 rather than 0; it doesn't really matter. What the definition is saying is that we can lay out the members of a sequence in a list with a first member, second member and so on. If $a : \mathbb{N} \to \mathbb{R}$, we usually write a_1, a_2 and so on, instead of the more formal $a(1)$, $a(2)$, even though we usually write functions in this way.

2.1.1 Examples of sequences

The most obvious example of a sequence is the sequence of natural numbers. Note that the integers are not a sequence, although we can turn them into a sequence in many ways; for example by enumerating them as 0, 1, -1, 2, $-2 \ldots$. Here are some more sequences:

Definition	First 4 terms	Limit
$a_n = n - 1$	0, 1, 2, 3	does not exist ($\to \infty$)
$a_n = \dfrac{1}{n}$	$1, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}$	0
$a_n = (-1)^{n+1}$	$1, -1, 1, -1$	does not exist (the sequence oscillates)
$a_n = (-1)^{n+1}\dfrac{1}{n}$	$1, -\dfrac{1}{2}, \dfrac{1}{3}, -\dfrac{1}{4}$	0
$a_n = \dfrac{n-1}{n}$	$0, \dfrac{1}{2}, \dfrac{2}{3}, \dfrac{3}{4}$	1
$a_n = (-1)^{n+1}\left(\dfrac{n-1}{n}\right)$	$0, -\dfrac{1}{2}, \dfrac{2}{3}, -\dfrac{3}{4}$	does not exist (the sequence oscillates)
$a_n = 3$	3, 3, 3, 3	3

A sequence doesn't have to be defined by a sensible "formula". Here is a sequence you may recognise:-

$$3, \quad 3.1, \quad 3.14, \quad 3.141, \quad 3.1415, \quad 3.14159, \quad 3.141592\ldots$$

where the terms are successive truncates of the decimal expansion of π.

Of course we can graph a sequence, and it sometimes helps. In Fig 2.1 we show a sequence of locations of (just the x coordinate) of a car driver's eyes. The interest is whether the sequence oscillates predictably.

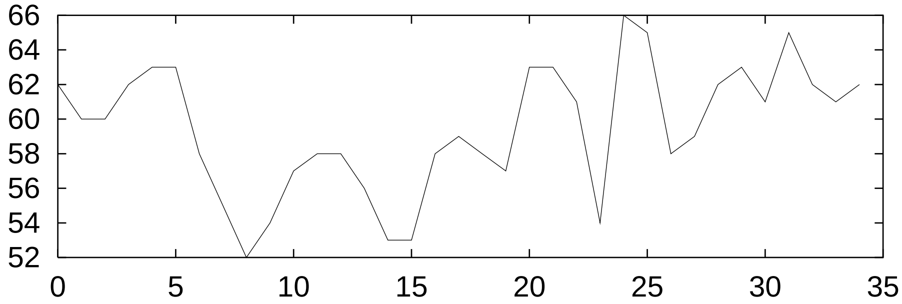

Figure 2.1: A sequence of eye locations.

Usually we are interested in what happens to a sequence "in the long run", or what happens "when it settles down". So we are usually interested in what happens when $n \to \infty$, or in the **limit** of the sequence. In the examples above this was fairly easy to see.

Sequences, and interest in their limits, arise naturally in many situations. One such occurs when trying to solve equations numerically; in Newton's method, we use the standard calculus approximation, that

$$f(a+h) \approx f(a) + h.f'(a).$$

If now we almost have a solution, so $f(a) \approx 0$, we can try to perturb it to $a+h$, which is a true solution, so that $f(a+h) = 0$. In that case, we have

$$0 = f(a+h) = f(a) + h.f'(a) \quad \text{and so} \quad h \approx \frac{f(a)}{f'(a)}.$$

Thus a better approximation than a to the root is $a + h = a - f(a)/f'(a)$.

If we take $f(x) = x^3 - 2$, finding a root of this equation is solving the equation $x^3 = 2$, in other words, finding $\sqrt[3]{2}$ In this case, we get the sequence defined as follows

$$a_1 = 1 \text{while} a_{n+1} = \frac{2}{3}a_n + \frac{2}{3a_n^2} \quad \text{if } n > 1. \tag{2.1}$$

Note that this makes sense: $a_1 = 1$, $a_2 = \dfrac{2}{3}.1 + \dfrac{2}{3.1^2}$ etc. Calculating, we get $a_2 = 1.333$, $a_3 = 1.2639$, $a_4 = 1.2599$ and $a_5 = 1.2599$ In fact the sequence does converge to $\sqrt[3]{2}$; by taking enough terms we can get an *approximation* that is as accurate as we need. [You can check that $a_5^3 = 2$ to 6 decimal places.]

Note also that we need to specify the accuracy needed. There is no *single* approximation to $\sqrt[3]{2}$ or π which will always work, whether we are measuring a flower bed or navigating a satellite to a planet. In order to use such a sequence of approximations, it is first necessary to specify an acceptable accuracy. Often we do this by specifying a neighbourhood of the limit, and we then often speak of an ϵ - neighbourhood, where we use ϵ (for error), rather than δ (for distance).