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Analysis I

Integral Representations and Asymptotic Methods

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Contents

I. Series and Integral Representations

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Introduction

Infinite series, and their analogues—integral representations, became fundamental tools in mathematical analysis, starting in the second half of the seventeenth century. They have provided the means for introducing into analysis all of the so-called transcendental functions, including those which are now called elementary (the logarithm, exponential and trigonometric functions). With their help the solutions of many differential equations, both ordinary and partial, have been found. In fact the whole development of mathematical analysis from Newton up to the end of the nineteenth century was in the closest way connected with the development of the apparatus of series and integral representations. Moreover, many abstract divisions of mathematics (for example, functional analysis) arose and were developed in order to study series.

In the development of the theory of series two basic directions can be singled out. One is the justification of operations with infinite series, the other is the creation of techniques for using series in the solution of mathematical and applied problems. Both directions have developed in parallel. Initially progress in the first direction was significantly smaller, but, in the end, progress in the second direction has always turned out to be of greater difficulty.

It would be a mistake to think that the justification of operations with series interested our predecessors less than us, or that they valued techniques more highly than rigour. Newton's proofs were completely rigorous, and he was reluctant to publish an insufficiently justified theory of fluxions. In my opinion, the small advances in the justification of operations with infinite series is explained by the absence of a suitable language in which to conveniently speak of these operations, and the creation of a language requires incomparably greater efforts than the proof of individual results. As a rule, the creation of a language is the work of several generations. In this respect we can refer to the example of Euler, whose research affected his contemporaries by its depth and non-triviality, but shocked them with its lack of rigour. To a modern reader the arguments of Euler do not seem to be so very non-rigorous. Simply, Euler already understood the principle of analytic continuation (for single-valued analytic functions), but the absence of a suitable language prevented him from transmitting this understanding to his contemporaries.

In the mid nineteenth century there was already a completely modern understanding of a convergent series which allowed one to prove the required results with complete rigour and to distinguish valid arguments from invalid ones. However, left over from the seventeenth and eighteenth centuries were many puzzling unjustified arguments which, for all their lack of justification, led to true results by significantly briefer routes. The expansion of the main points of these arguments and the creation of new means of justifying operations with divergent series and integrals was one of the basic achievements of the last century. A short account of the stages in the development of the modern approach to these questions forms the content of the first chapter of this article.

The second chapter is devoted to the second direction; techniques for using series and integral representations in mathematical analysis. The selection of the material for this chapter presented a most difficult problem, and the chosen solution is purely subjective. I have desisted from an attempt to list results, since this route would have required a much larger volume and would have ended only with the production of a reference book; completely useless for reading. A unique opportunity for me, it would appear, to give an exposition of fundamental methods. However, even this path has its own obstacles. The fact is that almost every method which has been used in analysis has generated, in its applications to different objects, extensive theories. Some of these theories have been successfully concluded, some are being rapidly developed and some have come to a dead end. In any of these cases a detailed story of these theories is inadvisable. I have decided to recount in this article only those analytic methods which have not yet been developed into a general theory. Almost all of it is around 100 years old (or more), but is familiar only to sophisticated analysts. To establish the authorship of these methods is most often impossible; they represent the birth of "mathematical folklore".

I have tried not to overburden the article with historical or bibliographical information (although the temptation in both directions was strong). In compiling the bibliography I have proceeded on the premise that its purpose is to assist the reader to quickly find the necessary sources (and not to display the erudition of the author). Therefore I have avoided references to obscure literature. If the reader wishes he can find exotic references in the bibliographies of the books quoted.

Chapter 1
The Evolution of the Concept of Convergence

§ 1. Numerical Series

The theory of convergence of numerical series assumed its completely modern form in the middle of the nineteenth century.[1] In the last 150 years there have been no new results and no new notations. We will now list the basic definitions and results.

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[1] It would be correct to say that at the beginning of the nineteenth century they began to speak of convergence of numerical series in a language close to the language of the textbooks of our time. The idea of convergence itself, apparently, was not that different from that contemporary with Ancient Greece, but to detach this notion from its method of expression is very difficult.

A *numerical series* is an infinite sum

$$\sum_{n=1}^{\infty} u_n = u_1 + u_2 + u_3 + \cdots \tag{1.1}$$

where the u_n are real or complex numbers. The number u_n is called the *general term of the series* and the number

$$s_n = \sum_{k=1}^{n} u_k = u_1 + \cdots + u_n$$

is called a *partial sum of the series.*

No real meaning,[2] in general, is imposed on the infinite sum (1.1). If there is a finite limit

$$s = \lim_{n\to\infty} s_n$$

then the symbolic notation (1.1) gains a meaning. In this case the series is called *convergent* and s is called the *sum of the series.*

The Cauchy criterion. A numerical series (1.1) is convergent if and only if for each $\varepsilon > 0$ there is a number $N(\varepsilon)$ such that for all $n > N(\varepsilon)$ and for all $m \geqq 0$ the inequality

$$|u_n + \cdots u_{n+m}| < \varepsilon$$

is satisfied.

A series is called *absolutely convergent* if the series

$$\sum_{n=1}^{\infty} |u_n|$$

is convergent.

From the Cauchy criterion it is clear that:

Each absolutely convergent series is convergent.

If a series converges then its general term tends to zero.

Although the idea of convergence of a series was precisely formulated only at the beginning of the nineteenth century the majority of the tests for convergence were found somewhat earlier. We list the basic convergence tests, beginning with tests for absolute convergence.

All the tests for absolute convergence rest on the so-called *comparison test*:

Let $a_1 + a_2 + \cdots$ be a convergent series with non-negative terms. If the general term of (1.1) satisfies

$$|u_n| \leqq a_n, \qquad n = 1, 2, \ldots,$$

then the series (1.1) converges absolutely.

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[2] A characteristic example of the slowness of change in the language of mathematicians. Even in modern terminology there are still traces of lost beliefs. The definition given implicitly endows any series (regardless of its convergence) with some value. In the seventeenth century it was firmly believed that each series had a definite sum, although we might not know a method of finding it.

The simplest infinite series, whose convergence was well-known even in antiquity, is the geometric progression with a multiplier less than one. Comparison with a geometric progression gives us Cauchy's test:

If

$$\limsup_{n\to\infty} (|u_n|)^{1/n} < 1,$$

then the series (1.1) converges absolutely.

D'Alemberts test is obtained by the same route:

If

$$\lim_{n\to\infty} \left|\frac{u_{n+1}}{u_n}\right| < 1,$$

then the series (1.1) converges absolutely.

Cauchy's test is essentially stronger than D'Alembert's test, but the latter is rather more convenient to apply when the general term of the series is in the form of products and quotients of factorials.

In order to obtain more precise absolute convergence tests by means of the comparison test it is necessary to have a larger stock of convergent series. This stock has been obtained via the integral calculus. With its help the following test has been obtained, known as the Cauchy integral test:

Let a positive function $f(x)$ be continuous for $x \geqq a$ and monotonely tending to zero as $x \to +\infty$. If

$$\lim_{x\to+\infty} \int_a^x f(t)\,dt < \infty,$$

then the series

$$f(a) + f(a+1) + f(a+2) + \cdots \tag{1.2}$$

converges absolutely, and if

$$\lim_{x\to+\infty} \int_a^x f(t)\,dt = +\infty,$$

then the series (1.2) diverges.

For

$$f(x) = x^{-\alpha}, \qquad f(x) = \frac{(\log x)^{-\alpha}}{x}, \qquad f(x) = \frac{(\log(\log x))^{-\alpha}}{x \log x}, \ldots$$

the integrals can be calculated, and the Cauchy integral test gives us a scale of comparisons sufficient for the majority of problems. Namely:

the series

$$\sum_{n=1}^{\infty} n^{-\alpha}, \qquad \sum_{n=2}^{\infty} \frac{(\log n)^{-\alpha}}{n}, \qquad \sum_{n=3}^{\infty} \frac{(\log(\log n))^{-\alpha}}{n \log n}, \ldots$$

converge for $\alpha > 1$ and diverge for $\alpha \leqq 1$.

In the nineteenth century many other tests for absolute convergence were devised but at present have been forgotten as unnecessary.[3]

Convergent but not absolutely convergent series are called *conditionally convergent*. If absolutely convergent series are practically no different to finite sums, then conditionally convergent series require a very much more careful treatment, as is shown, for example, by the following result.

Riemann's theorem. *By varying the enumeration of the terms of a conditionally convergent series (with real terms) it is possible to obtain a series which converges to a preassigned sum, or even a divergent series.*

A similar result holds for series with complex terms but the sum of the new series may either be any point of the complex plane, or any point on some line in the complex plane.[4]

There are comparatively few tests for conditional convergence of series which do not reduce to tests for its absolute convergence. The most general is *Abel's test*:

Let a_n and b_n $(n = 1, 2, \ldots)$ be two sequences of complex numbers, having the properties

$$\left|\sum_{k=1}^{n} a_k\right| \leqq M < \infty, \qquad n = 1, 2, \ldots,$$

and

$$\sum_{n=1}^{\infty} |b_{n+1} - b_n| < \infty.$$

Then the series

$$\sum_{n=1}^{\infty} a_n b_n$$

converges (generally speaking, conditionally).

The best known is Leibniz's test, which is obtained from Abel's test when $a_n = (-1)^n$ and b_n is a positive sequence monotonely converging to zero as $n \to \infty$.

§ 2. Improper Integrals

The concept of an improper integral was formulated in the nineteenth century and to this day is used in elementary textbooks on mathematical analysis, however, from the modern point of view, it has lost its significance. We will say

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[3] A reader wishing to familiarise himself with the ancient tests for convergence may see [9, 30].

[4] This analogue of Riemann's theorem for series with complex terms is set as a problem in Pólya and Szegö [44] without reference to authorship.

more about this later, but first we will give the definition of the concept and list the basic tests for convergence.

Let a complex-valued function $f(x)$ be Riemann integrable in each interval (a, b'), where $a < b' < b$, but not (Riemann) integrable in (a, b). Then we may speak of an *improper integral* of $f(x)$ on (a, b).[5] If the limit of the integrals of $f(x)$ over (a, b') exists as $b' \to b$, $b' < b$, then we say that the *improper integral converges* at the point b.

Similarly we define an improper integral convergent (or not) at the lower limit of integration.

An *improper integral* of a function $f(x)$ over a given interval is called *absolutely convergent* if the improper integral of $|f(x)|$ is convergent over the same interval.

Tests for convergence of improper integrals are not very different to the tests for convergence of numerical series.

As for series the basic test is the *comparison test*:

Let $g(x)$ be non-negative on (a, b) and let the integral of $g(x)$ over (a, b) converge. If $|f(x)| \leqq g(x)$ holds on (a, b) and $f(x)$ is integrable on each interior interval, then the improper integral of $f(x)$ on (a, b) is absolutely convergent.

A scale of convergent and divergent integrals is constructed even more easily than for series since integrals are more easily evaluated. We will give a scale of convergence in two versions; one for $a = 0$ the other for $b = +\infty$.

We introduce the notation

$$\log_1 x = \log x; \qquad \log_k x = \log(\log_{k-1} x), \qquad k = 2, 3, \ldots$$

If a function $f(x)$, which is integrable on each interval (a, c), satisfies for sufficiently large x the inequality

$$|f(x)| \leqq M \frac{(\log_k x)^{-\alpha}}{x \log_1 x \ldots \log_{k-1} x}$$

for some positive integer k and some $\alpha > 1$, then the improper integral of $f(x)$ over $(a, +\infty)$ is absolutely convergent (at infinity).

If a function $f(x)$, which is integrable on each interval (c, b) with $0 < c < b$, satisfies for sufficiently small x the inequality

$$|f(x)| \leqq M \frac{(\log_k(1/x))^{-\alpha}}{x \log_1(1/x) \ldots \log_{k-1}(1/x)}$$

for some positive integer k and some $\alpha < 1$, then the improper integral of $f(x)$ over $(0, b)$ is absolutely convergent (at zero).

For conditional convergence of improper integrals also there is a test similar to Abel's test for series.

Let the functions $g(x)$ and $h(x)$ be given on $(a, +\infty)$, moreover let $g(x)$ be positive and monotonely tending to zero as $x \to +\infty$ and let $h(x)$ satisfy the

[5] For $b = \infty$ Hardy named these integrals "infinite".

condition

$$\left|\int_a^x h(t)\,dt\right| \leqq M < \infty, \qquad x \geqq a.$$

Then the improper integral of the function $f(x) = g(x)h(x)$ on $(a, +\infty)$ converges (at infinity).

The essential difference between series and improper integrals is the absence of a simple necessary condition for convergence of the integral (similar to the general term of a convergent series tending to zero). In particular, in an integral absolutely convergent at infinity, the integrand need not tend to zero. As an example take the function

$$f(x) = \sum_{n=1}^{\infty} n^m \exp[-2^{2n}(x-n)^2\}, \qquad m > 0.$$

It is easily verified that the integral of this function over the whole line converges (and, moreover, absolutely since $f(x)$ is positive). At the same time it is easy to see that

$$f(n) > n^m, \qquad n = 1, 2, \ldots$$

We have already mentioned at the beginning of this section that from the modern viewpoint the idea of an improper integral has lost its significance. Here we must distinguish between absolute and conditional convergence.

First we will discuss the question of absolute convergence of improper integrals.

In modern mathematics the Riemann integral has for a long time given way to the Lebesgue integral. If we consider an improper Riemann integral, then its absolute convergence is a corollary of the existence of the integral as a Lebsgue integral. However, in defence of the classical heritage it is worth saying that we deal rather with a terminological improvement. In fact, the question of existence of the Lebesgue integral of a measurable function easily reduces to a question of absolute convergence of an improper Riemann integral. Namely, suppose we have a Lebesgue integral

$$I = \int_D f(x_1, \ldots, x_n)\,dv \tag{1.3}$$

where D is a domain in $\mathbf{R}^n$ and dv is the volume element in $\mathbf{R}^n$. According to the definition, the integral (1.3) exists if and only if the integral

$$I^* = \int_D |f(x_1, \ldots, x_n)|\,dv \tag{1.3*}$$

exists (also in the Lebesgue sense). According to a well-known formula from Lebesgue integration theory

$$I^* = \int_0^\infty t\mu(t)\,dt \tag{1.4}$$

where $\mu(t)$ is the Lebesgue measure of the set of points of D for which

$$|f(x_1, \ldots, x_n)| \geqq t.$$

The function $\mu(t)$ is non-negative and non-increasing. Therefore the function $t\mu(t)$ is Riemann integrable in each (a, b) with $a > 0$ and $b < +\infty$, and (1.4) is then an improper Riemann integral. Its convergence (at both limits) is equivalent to the existence of the integral (1.3*) and hence (1.3).

Thus, although in passing to the Lebesgue integral the notion of absolute convergence loses its significance, the tests for absolute convergence remain useful for research into the question of existence of the integral.

The replacement of the concept of an absolutely convergent Riemann integral by the concept of existence of a Lebesgue integral is particularly convenient when the question concerns multiple integrals. The fact is that the definition given by us at the beginning of this section does not generalise well to the many-dimensional case.

The position with conditional convergence of integrals is noticeably more complex. The discussion of many-dimensional integrals suggests that conditional convergence is far from the best means to attach a meaning to a non-existent integral. In fact, for many-dimensional integrals, the significantly more convenient concept of the principal value of an integral is widely used. We will give the definition of this concept in one of the simplest cases.

Let the function $f(x_1, \ldots, x_n)$ be continuous on the closure of a domain $D \subset \mathbf{R}^n$, with the exception of one point $(x_1^0, \ldots, x_n^0)$. Denote by D_ε the domain obtained from D by removing the ball

$$|x - x_1^0|^2 + \cdots + |x - x_n^0|^2 < \varepsilon^2.$$

If the limit

$$\lim_{\varepsilon \to 0} \int_{D_\varepsilon} f(x_1, \ldots, x_n)\, dv$$

exists, then it is called the *principal value*[6] *integral* of $f(x_1, \ldots, x_n)$ over D. The principal value integral is usually denoted

$$\mathrm{P} \int_D f(x_1, \ldots, x_n)\, dv.$$

Both conditionally convergent integrals and principal value integrals are rather feeble attempts to attach a definite value to a non-existent integral. In the following sections we will speak of much more drastic measures taken in this direction.

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[6] The concept of principal value integral was introduced by Cauchy in the first half of the nineteenth century but only became widely used in the twentieth century. The basic works on this concept are [26] and [46]. For a detailed survey and bibliography see [8].

§ 3. Regular Methods of Summation

In the seventeenth and even the eighteenth centuries mathematicians believed that each series (convergent or divergent) had a well-defined sum; for convergent series it was possible to find it simply by successively adding the terms, whereas for divergent series more complicated methods were needed. Almost nobody doubted the correctness of the formulae

$$1 - 1 + 1 - 1 + \cdots = 1/2$$

$$1 - 2 + 3 - 4 + \cdots = 1/4,$$

since the sums of these series, when computed by various methods were the same. The sum of the series

$$1 - 1! + 2! - 3! + \cdots$$

was calculated by Euler to three decimal places using a method which we now call the Euler summation method.

One of the most widely applied methods of evaluating sums of series is the following.

First a certain number of partial sums of the series are calculated. If the sums start to coincide (to the desired accuracy) after some index, then the value obtained is regarded as the sum of the series. If the values of the partial sums continue to noticeably diverge, then their arithmetic means are calculated and if they start to coincide after some index, then the sum of the series is taken to be the value obtained. Often the arithmetic means of the arithmetic means are taken.

The practical utility of the method described is unquestionable. In fact, the calculation of each successive term of a series is a fairly complicated problem. Managing without these calculations at the expense of calculating the arithmetic means of values already found meant containing a great saving in time.

Euler's method is close in spirit to that just described but is more refined. To wit, given a series $u_1 + u_2 + u_3 + \cdots$ Euler constructed a new series $v_1 + v_2 + v_3 + \cdots$ as follows

$$v_1 = \frac{1}{2}u_1, \qquad v_n = \frac{1}{2^n}\left(u_1 + \binom{n-1}{1}u_2 + \cdots + \binom{n-1}{n-1}u_n\right), \qquad n = 2, 3, \ldots$$

where $\binom{n}{k}$ are the binomial coefficients. Having written down this calculation scheme it is not difficult to see that the partial sums of the new series are not much more difficult to calculate than in the method of arithmetic means. At the same time Euler's method, in many cases, gave significantly greater acceleration of the convergence.

Both the method of arithmetic means and Euler's method can be applied with equal success to convergent and divergent series. It is easy to see that if the original series converges then both the methods lead to the sum. However, these methods give determinate values even for the sums of many divergent series. In the nineteenth century a great many methods of summation were proposed (the invention of new methods of summation stopped with the advent of the twentieth century).[7] The first fundamentally new step in the theory of summation methods was made in 1911 by Toeplitz.[8] He proved a theorem describing all the regular methods of summation—summation methods having the natural properties of linearity and taking convergent series to convergent series.

It is more convenient to formulate Toeplitz's theorem in terms of sequences rather than series. To apply the theorem to series the sequence of partial sums is taken as the sequence.

Let

$$A = (a_{nk}), \qquad n, k = 1, 2, \ldots, \qquad k \leqq n,$$

be an infinite triangular matrix. The matrix A associates with each sequence $\{s_n\}$ the sequence $\{s_n^A\}$, where

$$s_n^A = \sum_{k=1}^{n} a_{nk} s_k, \qquad n = 1, 2, \ldots$$

We will say that $\{s_n\}$ is *summable* with sum *s by the method defined by A*, if

$$\lim_{n\to\infty} s_n^A = s.$$

The *summation method* defined by A is called *regular* if $s_n \to s$ implies $s_n^A \to s$. *Toeplitz's theorem* is as follows.

A summation method defined by a matrix A is regular if and only if the following two conditions are satisfied,

$$\sum_{k=1}^{n} a_{nk} = 1, \qquad n = 1, 2, \ldots,$$

and

$$\lim_{n\to\infty} a_{nk} = 0, \qquad k = 1, 2, \ldots$$

Summation methods defined by Toeplitz matrices include most of the known summation methods. For example, *the method of arithmetic means* corresponds to the matrix A with

$$a_{nk} = 1/n, \qquad n, k = 1, 2, \ldots, \qquad k \leqq n,$$

and *Euler's method* corresponds to the matrix with

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[7] A detailed survey of summation methods, containing the proofs of almost all the results and a great deal of interesting historical information, is given in [27].

[8] Toeplitz's original paper was published in an inaccessible Polish journal, but his results are presented in detail in many books, for example, [27] or [58].

$$a_{nk} = \frac{1}{2^n}\binom{k}{n}, \qquad n, k = 1, 2, \ldots, \qquad k \leqq n.$$

Nevertheless, certain summation methods widely used in analysis do not fall within the Toeplitz scheme; for example, the *Abel-Poisson method.* This method attaches to the series $u_1 + u_2 + u_3 \ldots$ the sum

$$s = \lim_{x \to 1-0} \sum_{n=1}^{\infty} u_n x^n.$$

This defect was eliminated almost immediately by Steinhaus.[9]

The simplicity of Toeplitz's criterion made it easy to construct regular summation methods with fairly unusual properties. For example, it turned out to be possible to construct regular methods which summed the series $1 - 1 + 1 - 1 + \cdots$ to any preassigned number. It could be said that Toeplitz's theorem struck a decisive blow at the naive belief in the existence of a definite sum for each series.

A very interesting subclass of Toeplitz summation methods are the *Hausdorff means*, considered by Hausdorff.[10] These methods are defined by matrices A of the form $A = \delta\mu\delta$, where μ is a diagonal matrix with positive diagonal elements, and

$$\delta = \left((-1)^k \binom{n}{k}\right), \qquad n, k = 1, 2, \ldots, \qquad n \geqq k.$$

For Hausdorff means interesting answers to the question of comparative strengths of methods and their consistency have been obtained. The research into Hausdorff means completed the destruction of the naive belief mentioned above.

It appears that Euler already understood the necessity to be aware of the "parentage" of a numerical series in order to sum it properly. In fact, Euler almost always dealt with power series, and the methods of summation he applied were by analytic continuation of power series. Thus, for example, the above-mentioned Euler method of summation reduces to regarding the series $u_1 + u_2 + u_3 + \cdots$ as the value of the power series

$$\sum_{n=1}^{\infty} u_n x^n$$

at $x = 1$; to calculate this value the series is expanded in powers of $y = \dfrac{x}{1+x}$ and the value of the new series at $y = \frac{1}{2}$ is taken. Many of the manipulations carried out by Euler stopped being mysterious after the creation, by Riemann and Weierstrass, of analytic function theory.

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[9] Steinhaus' result, published in the same journal issue as Toeplitz's result, was obtained on the basis of Toeplitz's result and using the same method. Therefore the more general Steinhaus result is often called Toeplitz's theorem.

[10] The summation methods named Hausdorff means were introduced not by Hausdorff but by Hurwitz. They are named after Hausdorff because he studied them in depth in a series of papers [28] and [29]. One of the chapters of [27] is devoted to an account of Hausdorff's theory.

It is worth stressing that the majority of summation methods have been developed, not out of an abstract desire to find a sum for any divergent series, but for the summation of divergent series which arise in concrete problems. We will now give two results of this kind.

Let the power series

$$f(z) = \sum_{n=1}^{\infty} f_n z^n$$

converge in some neighbourhood of $z = 0$. The *Mittag-Leffler star* of the series $f(z)$ is the domain D_f consisting of points to which the series can be continued along line segments passing through $z = 0$. D_f is a simply-connected domain and the result of continuing $f(z)$ to D_f is a single-valued function which we denote by $F(z)$.

The limit

$$\lim_{\delta \to +0} \sum_{n=0}^{\infty} \frac{f_n z^n}{\Gamma(1 + n\delta)}$$

exists for each $z \in D_f$ and is equal to $F(z)$.

This result is due to Mittag-Leffler.[11]

As a second result we quote *Fejer's theorem.*[12]

The Fourier series of any piecewise continuous function $\phi(x)$ sums by the method of arithmetic means to $\frac{1}{2}\{\phi(x + 0) + \phi(x - 0)\}$.

Summation methods have been discussed not just for series but also for improper integrals. The generalizations of the formulae present no special problems. For example, the summation formula for a series $u_1 + u_2 + \cdots$ by the method of arithmetic means takes the form

$$s = \lim_{n \to \infty} \sum_{k=1}^{n} \left(1 - \frac{k}{n}\right) u_k,$$

for summation of the integral

$$\int_a^{\infty} f(x)\, dx, \qquad a > 0$$

it takes the form

$$s = \lim_{t \to +\infty} \int_a^t \left(1 - \frac{x}{t}\right) f(x)\, dx.$$

The fundamental results on summation of integrals are concerned with the theory of Fourier integrals, where theorems analogous to the above theorem of Fejer have been proved.[13]

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[11] There is an account of this result in [27].

[12] See the original article by Fejer [20]. An account of Fejer's theorem and its various generalizations can be found in any book on trigonometric series.

[13] Many results on the summation of Fourier integrals are given in [50] and [56].

Toeplitz's theorem has also been generalized to summation methods for integrals. In this case the role of the matrix is played by a function of two variables.[14]

§4. Function Series

At the foundation of infinitesimal calculus is the systematic utilization, not of numerical series, but of function series. However, although the concept of convergence of a numerical series had taken a completely modern form by the turn of the nineteenth century, the development of the modern concepts of convergence of function series required another complete century. We will briefly tell the story of the formation of these ideas.

The meaning of convergence for a numerical series is fairly obvious—a convergent numerical series must have a sum which can be calculated to any degree of accuracy from its sequence of partial sums. A *function series* can, certainly, be considered as the collection of numerical series associated with all possible values of the variables and we can say that the *function series is convergent* if all of these numerical series converge. Such an approach has turned out to be unsatisfactory in many respects. The fundamental deficiency is that it severely restricts the role of the function series in analysis. The basic advantage of function series (widely used from the very beginnings of infinitesimal calculus) is that it is easy to perform many formal operations with these series—they can be added, multiplied, integrated, etc. The convergence of the series which occur in the intermediate steps is of no consequence. All that counts is the validity of the final result. Thus the notion of convergence of a function series must be aimed at providing a foundation for formal operations with function series.

The first serious results on function series were obtained by Weierstrass in the mid nineteenth century. He introduced a notion of convergence for function series which was stricter than convergence at each point. This notion, which was named *uniform convergence*, rapidly gained universal recognition.[15] It has come down to the elementary analysis courses of our time with practically no change. The fundamental theorems on uniformly convergent series are.

a) The sum of a uniformly convergent series of continuous functions is a continuous function.

b) A uniformly convergent series of continuous functions can be integrated term by term.

[14] These results are also presented in [27].

[15] Although Weierstrass had used the idea of uniform convergence by 1841, formally he was not the author of the idea. The definition was published in 1847 independently by Seidel and Stokes. The notion, nevertheless, has been associated with Weierstrass who willingly explained it in lectures and other verbal communications, but, in the words of Klein, he had an aversion to printers ink and did not like to publish his work.

These theorems now occupy an honourable place in analysis textbooks and are called the *Weierstrass theorems.*

We must mention also a simple test for uniform convergence called the *Weierstrass test.*

c) If functions $f_n(x)$, which are continuous in a domain D, satisfy the inequalities $|f_n(x)| \leqq u_n$ in this domain and the numerical series with general term u_n converges, then the function series with general term $f_n(x)$ converges uniformly in D.

Using these theorems it became possible to accurately justify the foundations of operations with uniformly convergent function series. However, the solution was by no means completely satisfactory—the requirement of uniform convergence of all intermediate series was clearly excessive. We pass over the struggle with the severity of this requirement.

In the nineteenth century mathematicians studied mainly two types of function series—the power series and the Fourier series (including Fourier integrals). These series are essentially different, and we discuss them separately.

In questions concerning power series more or less complete answers are given by the following two *theorems of Abel:*[16]

1. If a power series

$$\sum_{n=0}^{\infty} a_n z^n \tag{1.5}$$

converges at a point $z = z_0$ and $z_0 \neq 0$, then it converges uniformly in the disc $|z| \leqq r$, for any $r < |z_0|$.

2. Under the assumptions of the preceding theorem, the power series (1.5) converges uniformly on the line segment joining $z = 0$ to $z = z_0$.

The first Abel theorem implies the existence of a number R, $R \geqq 0$, such that (1.5) converges for $|z| < R$ and diverges for $|z| > R$. This number is called the *radius of convergence of the series* (1.5) and the disc $|z| < R$ is called the *disc of convergence* (the latter name is used only when $R > 0$).

Yet Euler, who carried out a variety of operations with power series, often operated with them outside the disc of convergence. From the point of view of uniform convergence such operations were totally illegitimate, but Euler obtained correct results with their help (unfortunately, operations almost the same as Euler's led other mathematicians to incorrect results). In clarifying his astonishing operations with divergent series Euler said that he was working not on the series, but on the functions which could be expanded into these series. The possibility of free transition from series to functions and conversely was guaranteed by the uniqueness of the expansion of a function into a power series.[17]

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[16] Abel's formulation has been somewhat modernised. In 1826, when he published his theorems, the notion of uniform convergence did not exist.

[17] Books on the history of mathematics do not mention who first noted the uniqueness of the expansion of an analytic function into a power series. This fact was considered selfevident long before the origin of a clear notion of a function.

The theory of analytic functions created by Weierstrass allowed many of Euler's arguments to be justified. We briefly explain the essence of the idea of analytic continuation which lies at the foundation of the Weierstrass theory.

A function expanded in a power series in a neighborhood of each point at which it is defined, was for a long time the principal object of study in mathematical analysis. It was thought at first that other functions simply did not occur; but when the existence of others was recognised, these *functions* came to be called *analytic*. It is easy to deduce the following result from the uniqueness of the power series expansion.

The principle of analytic continuation. *Let $f(z)$ be a function analytic in a domain D, and let G be a domain containing D. Then there is at most one function $F(z)$ analytic in G and coinciding with $f(z)$ in D.*

The function $F(z)$ is called the *analytic continuation* of $f(z)$ from D to G.

From the principle of analytic continuation it follows, in particular, that an analytic function is completely determined by its values in an arbitrarily small neighbourhood of a point at which it is analytic. Therefore it is natural to speak of the analytic continuation of an analytic function from one point to another. Unfortunately the definition says nothing about the properties of the continuation.

Weierstrass carried out a deep analysis of the notion of analytic continuation. He showed that analytic continuation from one point to another (if it was possible) reduced to a finite number of elementary operations, consisting of the expansion of a power series

$$\sum_{n=0}^{\infty} c_n(z-a)^n \tag{1.6}$$

into powers of $(z-b)$, where b is a point inside the disc of convergence of (1.6). The result of continuation depends, in general, on the choice of intermediate points or, what is the same thing, on the choice of curve along which we proceed from one point to another.

Weierstrass' research allows us to understand the essence of the phenomenon of multivalency of analytic functions, which was a fundamental source of error in the clumsy attempts at analytic continuation. Weiestrass' research on analytic function theory was developed further by Riemann. After Weierstrass and Riemann analytic continuation became a standard tool in the hands of analysts.

Thus, research into power series in the nineteenth century led to the development of analytic function theory and the uncovering of the mystery of analytic continuation.

Essentially different problems arose in the analysis of Fourier series and Fourier integrals.

The trigonometric series

$$\sum_{n=-\infty}^{\infty} f_n e^{inx} \tag{1.7}$$

is called the *Fourier series* of a function $f(x)$, integrable on $(-\pi, \pi)$, if the formulae

$$f_n = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}\,dx, \qquad n = 0, \pm 1, \pm 2, \ldots, \tag{1.8}$$

known as the *Fourier formulae*, hold.

For *Fourier integrals* slightly different terminology is currently accepted.

Let the function $f(x)$ be defined on $\mathbf{R}^n$. Then the integral

$$\tilde{f}(\xi) = (2\pi)^{-n/2}\int_{R^n} f(x)e^{i\cdot(x\xi)}\,dx, \tag{1.9}$$

where

$$(x\xi) = x_1\xi_1 + \cdots + x_n\xi_n, \qquad dx = dx_1\ldots dx_n,$$

is called the *Fourier transform* of $f(x)$. The formula

$$f(x) = (2\pi)^{-n/2}\int_{R^n} \tilde{f}(\xi)e^{-i\cdot(x\xi)}\,d\xi, \tag{1.10}$$

is called the *inversion formula for the Fourier transform*. Formulae (1.9) and (1.10) are also called *Fourier formulae*.

Trigonometric series occurred in mathematics long before Fourier, but only after his work did they attract the serious attention of mathematicians. This was because in his research into heat diffusion Fourier showed that trigonometric series could be a powerful tool for the solution of partial differential equations (the *Fourier method*).

Fourier's work provided a major stimulus for the study of divergent series and integrals. Fourier's operations with series and integrals were, from the viewpoint of the Weierstrass theory of uniform convergence, totally illegitimate, but the results obtained using these operations aroused no doubt whatsoever. As a typical example we quote the derivation of *Parseval's formula*.

Let

$$f(x) = \sum_{n=-\infty}^{\infty} f_n e^{inx}, \qquad g(x) = \sum_{n=-\infty}^{\infty} g_n e^{inx}.$$

Then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)g(x)\,dx = \sum_{n=-\infty}^{\infty} f_n g_n.$$

To derive this formula the series for $f(x)$ and $g(x)$ are multiplied out and integrated term by term, after which the elementary equalities

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i(m-n)x}\,dx = \delta_{m,n}, \qquad m, n = 0, 1, \ldots,$$

are used, where $\delta_{m,n} = \begin{cases} 0 & m \neq n, \\ 1 & m = n. \end{cases}$

This derivation is justified by the Weierstrass theory if the Fourier series of $f(x)$ and $g(x)$ converge uniformly on the whole interval $[-\pi, \pi]$. However, as was established quite a long time ago, the demand for uniform convergence of Fourier series imposes an excessively high degree of smoothness on the functions $f(x)$ and $g(x)$. To present this requirement more clearly we will now give the basic *tests for convergence of Fourier series.*[18]

Dirichlet's test. If $f(x)$ is of *bounded variation* on $[-\pi, \pi]$ (that is is a difference of two monotone functions), then its Fourier series converges at each point to

$$\frac{f(x+0)+f(x-0)}{2}$$

Dini's test. If the integral

$$\int_0^\pi |f(x_0+t)+f(x_0-t)-2f(x_0)| \cot(t/2)\, dt$$

is finite, then the Fourier series of $f(x)$ at $x = x_0$ converges to $f(x_0)$.

These tests give sufficient conditions for convergence of a Fourier series, but these sufficient conditions are quite close to necessary. In the first half of the twentieth century many different examples were constructed in which the Fourier series converges as badly as possible. Amongst these an example due to A.N. Kolmogorov was particularly remarkable: the Fourier series of an integrable function which diverged at all points.[19]

The justification of operations on Fourier integrals was in an even more parlous state than the justification of operations on Fourier series. In the theory of Fourier integrals were circulating formulae such as

$$(2\pi)^{-n} \int_{R^n} e^{i\cdot(x\xi)}\, dx = \delta(\xi),$$

(where $\delta(\xi)$ is the so-called Dirac delta function). Many attempts were made to justify operations on Fourier series and Fourier integrals by using summation methods. However, the simplest and most effective route to this end lay in another direction.

§ 5. Convergence in Function Spaces

The first new step in the development of the modern trends in convergence of function series could be regarded as the following result.[20]

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[18] See [11] or [58].
[19] See [11] or [58].
[20] The original papers are [21] and [45]. An account can be found in any book on trigonometric series.