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Mines/Richman/Ruitenburg: A Course in Constructive Algebra

(continued after the list of symbols)

Werner Balser

Formal Power Series and Linear Systems of Meromorphic Ordinary Differential Equations

Springer

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Mathematics Subject Classification (1991): 34-02

Library of Congress Cataloging-in-Publication Data
Balser, Werner, 1946–
Formal power series and linear systems of meromorphic ordinary differential equations
Werner Balser.
p. cm. — (Universitext)
Includes bibliographical references and index.
ISBN 0-387-98690-1 (alk. paper)
1. Power series. 2. Differential equations, Linear.
3. Functions, Meromorphic. I. Title.
QA295.B325 1999
515'.2432—dc21 99-31350

Printed on acid-free paper.

Production managed by Robert Bruni; manufacturing supervised by Jeffery Taub.
Photocomposed copy prepared from the author's LaTeX files.
Printed and bound by R.R. Donnelley and Sons, Harrisonburg, VA.
Printed in the United States of America.

9 8 7 6 5 4 3 2 1

ISBN 0-387-98690-1 Springer-Verlag New York Berlin Heidelberg SPIN 10706616

Für meine verstorbenen Eltern,

für meine liebe Frau und unsere drei Söhne.

Preface

This book aims at two, essentially different, types of readers: On one hand, there are those who have worked in, or are to some degree familiar with, the section of mathematics that is described here. They may want to have a source of reference to the recent results presented here, replacing my text [21], which is no longer available, but will need little motivation to start using this book. So they may as well skip reading this introduction, or immediately proceed to its second part (p. xi) which in some detail describes the content of this book. On the other hand, I expect to attract some readers, perhaps students of colleagues of the first type, who are not familiar with the topic of the book. For those I have written the first part of the introduction, hoping to attract their attention and make them willing to read on.

Some Introductory Examples

What is this book about? If you want an answer in one sentence: It is concerned with formal power series – meaning power series whose radius of convergence is equal to zero, so that at first glance they may appear as rather meaningless objects. I hope that, after reading this book, you may agree with me that these formal power series are fun to work with and really important for describing some, perhaps more theoretical, features of functions solving ordinary or partial differential equations, or difference equations, or perhaps even more general functional equations, which are, however, not discussed in this book.

Do such formal power series occur naturally in applications? Yes, they do, and here are three simple examples:

1. The formal power series $\hat{f}(z) = \sum_0^\infty n!\, z^{n+1}$ formally satisfies the ordinary differential equation (ODE for short)
$$z^2\, x' = x - z. \qquad (1)$$
But everybody knows how to solve such a simple ODE, so why care about this divergent power series? Yes, that is true! But, given a slightly more complicated ODE, we can no longer *explicitly compute its solutions in closed form.* However, we may still be able to compute solutions in the form of power series. In the simplest case, the ODE may even have a solution that is a polynomial, and such solutions can sometimes be found as follows: *Take a polynomial $p(z) = \sum_0^m p_n\, z^n$ with undetermined degree m and coefficients p_n, insert into the ODE, compare coefficients, and use the resulting equations, which are linear for linear ODE, to compute m and p_n.* In many cases, in particular for large m, we may not be able to find the values p_n explicitly. However, we may still succeed in showing that the system of equations for the coefficients has one or several solutions, so that at least the existence of polynomial solutions follows. In other cases, when the ODE does not have polynomial solutions, one can still try to find, or show the existence of, solutions that are "polynomials of infinite degree," meaning power series
$$\hat{f}(z) = \sum_0^\infty f_n\,(z - z_0)^n,$$
with suitably chosen z_0, and f_n to be determined from the ODE. While the approach at first is very much analogous to that for polynomial solutions, two new problems arise: For one thing, we will get a system of infinitely many equations in infinitely many unknowns, namely, the coefficients f_n; and secondly, we are left with the problem of determining the radius of convergence of the power series. The first problem, in many cases, turns out to be relatively harmless, because the system of equations usually can be made to have the form of a *recursion*: Given the coefficients $f_0, \ldots, f_n$, we can then compute the next coefficient f_{n+1}. In our example (1), trying to compute a power series solution $\hat{f}(z)$, with $z_0 = 0$, immediately leads to the identities $f_0 = 0$, $f_1 = 1$, and $f_{n+1} = n\, f_n$, $n \geq 1$. Even to find the radius of convergence of the power series may be done, but as the above example shows, *it may turn out to be equal to zero!*

2. Consider the *difference equation*
$$x(z+1) = (1 - a\, z^{-2})\, x(z).$$

After some elementary calculations, one can show that this difference equation has a unique solution of the form $\hat{f}(z) = 1 + \sum_1^\infty f_n z^{-n}$, which is a power series in $1/z$. The coefficients can be uniquely computed from the recursion obtained from the difference equation, and they grow, roughly speaking, like $n!$ so that, as in the previous case, the radius of convergence of the power series is equal to zero. Again, this example is so simple that one can explicitly compute its solutions in terms of Gamma functions. But only slightly more complicated difference equations cannot be solved in closed form, while they still have solutions in terms of formal power series.

3. Consider the following problem for the heat equation:

$$u_t = u_{xx}, \quad u(0, x) = \varphi(x),$$

with a function φ that we assume holomorphic in some region G. This problem has a unique solution $u(t, x) = \sum_0^\infty u_n(x)\, t^n$, with coefficients given by

$$u_n(x) = \frac{\varphi^{(2n)}(x)}{n!}, \qquad n \geq 0.$$

This is a power series in the variable t, whose coefficients are functions of x that are holomorphic in G. As can be seen from Cauchy's Integral Formula, the coefficients $u_n(x)$, for fixed $x \in G$, in general grow like $n!$ so that the power series has radius of convergence equal to zero.

So formal power series do occur naturally, but what are they good for? Well, this is exactly what this book is about. In fact, it presents two different but intimately related aspects of formal power series:

For one thing, the very general theory of *asymptotic power series expansions* studies certain functions f that are holomorphic in a sector S but singular at its vertex, and have a certain *asymptotic behavior* as the variable approaches this vertex. One way of describing this behavior is by saying that the nth derivative of the function approaches a limit f_n as the variable z, inside of S, tends toward the vertex z_0 of the sector. As we shall see, this is equivalent to saying that the function, in some sense, is *infinitely often differentiable at z_0, without being holomorphic there, because the limit of the quotient of differences will only exist once we stay inside of the sector.* The values f_n may be regarded as the coefficients of *Taylor's series* of f, but this series may not converge, and even when it does, it *may not converge toward the function f*. Perhaps the simplest example of this kind is the function $f(z) = e^{-1/z}$, whose derivatives all tend to $f_n = 0$ whenever z tends toward the origin in the right half-plane. This also shows that, unlike for functions that are holomorphic at z_0, this Taylor series alone does not determine the function f. In fact, given any sector S, *every* formal power series $\hat{f}$ arises as an asymptotic expansion of *some* f that is holomorphic in S, but this f *never is uniquely determined by* $\hat{f}$, so that in particular the value of the function at a given point $z \neq z_0$ *in general cannot be computed from the asymptotic power series.* In this book, the theory of asymptotic power series expansions is presented, not only for the case when the coefficients are numbers, but also for series whose coefficients are in a given Banach space. This generalization is strongly motivated by the third of the above examples.

While general formal power series *do not determine one function*, some of them, especially the ones arising as solutions of ODE, are *almost as well-behaved as convergent ones*: One can, more or less explicitly, *compute some function* f from the divergent power series $\hat{f}$, which in a certain sector is asymptotic to $\hat{f}$. In addition, this function f has other very natural properties; e.g., it satisfies the same ODE as $\hat{f}$. This theory of *summability* of formal power series has been developed very recently and is the main reason why this book was written.

If you want to have a simple example of how to compute a function from a divergent power series, take $\hat{f}(z) = \sum_0^\infty f_n z^n$, assuming that $|f_n| \leq n!$ for $n \geq 0$. Dividing the coefficients by $n!$ we obtain a new series converging at least for $|z| < 1$. Let $g(z)$ denote its sum, so g is holomorphic in the unit disc. Now the general idea is to define the integral

$$f(z) = z^{-1} \int_0^\infty g(u)\, e^{-u/z}\, du \tag{2}$$

as the sum of the series $\hat{f}$. One reason for this to be a suitable definition is the fact that if we replace the function g by its power series and integrate termwise (which is illegal in general), then we end up with $\hat{f}(z)$. While this motivation may appear relatively weak, it will become clear later that this nonetheless is an excellent definition for a function f deserving the title *sum of* $\hat{f}$ – *except that the integral* (2) *may not make sense for one of the following two reasons:* The function g is holomorphic in the unit disc but may be undefined for values u with[1] $u \geq 1$, making the integral entirely meaningless. But even if we assume that g can be holomorphically continued along the positive real axis, its rate of growth at infinity may be such that the integral diverges. So you see that there are some reasons that keep us from getting a meaningful sum for $\hat{f}$ in this simple fashion, and therefore we shall have to consider more complicated ways of *summing* formal power series. Here we shall present a summation process, called *multisummability*, that can handle every formal power series which solves an ODE, but is still not general enough for solutions of certain difference equations or partial differential equations. *Jean Ecalle*, the founder of the theory of multisummability, has also outlined some more general

[1] Observe that such an inequality should be understood as saying: Here, the number u must be *real* and at least 1.

summation methods suitable for difference equations, but we shall not be concerned with these here.

Content of this Book

This book attempts to present the theory of linear ordinary differential equations in the complex domain from the new perspective of multisummability. It also briefly describes recent efforts on developing an analogous theory for *nonlinear systems, systems of difference equations, partial differential equations, and singular perturbation problems.* While the case of linear systems may be said to be very well understood by now, much more needs to be done in the other cases.

The material of the book is organized as follows: The first two chapters contain entirely classical results on the structure of solutions near regular, resp.[2] regular-singular, points. They are included here mainly for the sake of completeness, since none of the problems that the theory of multisummability is concerned with arise in these cases. A reader with some background on ODE in the complex domain may very well skip these and immediately advance to Chapter 3, where we begin discussing the *local theory of systems near an irregular singularity.* Classically, this theory starts with showing existence of formal fundamental solutions, which in our terminology will turn out to be *multisummable, but not k-summable, for any $k > 0$.* So in a way, these classical formal fundamental solutions are relatively complicated objects. Therefore, we will in Chapter 3 introduce a different kind of what we shall call *highest level formal fundamental solutions*, which have much better theoretical properties, although they are somewhat harder to compute. In the following chapters we then present the theory of asymptotic power series with special emphasis on Gevrey asymptotics and k-summability.

In contrast to the presentation in [21], we here treat *power series with coefficients in a Banach space.* The motivation for this general approach lies in applications to PDE and singular perturbation problems that shall be discussed briefly later. A reader who is not interested in this general setting may concentrate on series with coefficients in the complex number field, but the general case really is not much more difficult.

In Chapters 8 and 9 we then return to the theory of ODE and discuss the *Stokes phenomenon* of highest level. Here it is best seen that the approach we take here, relying on highest level formal fundamental solutions, gives a far better insight into the structure of the Stokes phenomenon, because it avoids mixing the phenomena occurring on different levels. Nonetheless, we then present the theory of multisummability in the following chapters and indicate that the classical formal fundamental solutions are indeed multisummable. The remaining chapters of the book are devoted to related but

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[2]Short for "respectively."

different problems such as *Birkhoff's reduction problem* or applications of the theory of multisummability to *difference equations, partial differential equations, or singular perturbation problems.* Several appendices provide the results from other areas of mathematics required in the book; in particular some well-known theorems from the theory of complex variables are presented in the more general setting of functions with values in a Banach space.

The book should be readable for students and scientists with some background in matrix theory and complex analysis, but I have attempted to include all the (nonelementary) results from these areas in the appendices. A reader who is mainly interested in the asymptotic theory and/or multisummability may leave out the beginning chapters and start reading with Chapters 4 through 7, and then go on to Chapter 10 – these are pretty much independent of the others in between and may be a good basis for a course on the subject of asymptotic power series, although the remaining ones may provide an excellent motivation for such a general theory to be developed.

Personal Remarks

Some personal remarks may be justified here: In fall of 1970, I came to the newly founded University of Ulm to work under the direction of *Alexander Peyerimhoff* in *summability theory.* About 1975 I switched fields and, jointly with *W. B. Jurkat* and *D. A. Lutz*, began my studies in the very classical, yet still highly active, field of *systems of ordinary linear differential equations whose coefficients are meromorphic functions of a complex variable* (for short: meromorphic systems of ODE). This field has occupied most of my (mathematical) energies, until almost twenty years later when I took up summability again to apply its techniques to the *divergent power series* that arise as *formal solutions* of meromorphic ODE. In this book, I have made an effort to represent the classical theory of meromorphic systems of ODE in the new light shed upon it by the recent achievements in the theory of summability of formal power series.

After more than twenty years of research, I have become highly addicted to this field. I like it so much because it gives us a splendid opportunity to obtain significant results using standard techniques from the theory of complex variables, together with some matrix algebra and other classical areas of analysis, such as summability theory, and I hope that this book may infect others with the same enthusiasm for this fascinating area of mathematics. While one may also achieve useful results using more sophisticated tools borrowed from advanced algebra, or functional analysis, such will not be required to understand the content of this book.

I should like to make the following acknowledgments: I am indebted to the group of colleagues at Grenoble University, especially *J. DellaDora, F. Jung*, and *M. Barkatou* and his students. During my appointment as *Pro-*

fesseur Invité in September 1997 and February 1998, they introduced me to the realm of computer algebra and helped me prepare the corresponding section, and in addition created a perfect environment for writing a large portion of the book while I was there. In March 1998, while I was similarly visiting Lille University, *Anne Duval* made me appreciate the very recent progress on application of multisummability to the theory of *difference equations*, for which I am grateful as well. I would also like to thank many other colleagues for support in collecting the numerous references I added, and for introducing me to related, yet different, applications of multisummability on formal solutions of partial differential equations and singular perturbation problems. Last, but not least, I owe thanks to my two teachers at Ulm University, *Peyerimhoff* (who died all too suddenly in 1996) and *Jurkat*, who were not actively involved in writing, but from whom I acquired the mathematics, as well as the necessary stamina, to complete this book.

Ulm, Germany, 1999 Werner Balser

Contents

1
Basic Properties of Solutions

In this first chapter, we discuss some basic properties of linear systems of ordinary differential equations having a coefficient matrix whose entries are holomorphic functions in some region G. A reader who is familiar with the theory of systems whose coefficient matrix is constant, or consists of continuous functions on a real interval, will see that all of what we say here for the case of a *simply connected region* G, i.e., a region "without holes," is quite analogous to the real-variable situation, but we shall discover a new phenomenon in case of multiply connected G. While for simply connected regions solutions always are holomorphic *in the whole region* G, this will no longer be true for multiply connected ones: Solutions will be *locally holomorphic*, i.e., holomorphic on every disc contained in G. Globally, however, they will in general be *multivalued* functions that should best be considered on some Riemann surface over G. As an example, observe that $x(z) = \sqrt{z(1-z)}$ is a solution of the equation

$$x' = \frac{1-2z}{2z\,(1-z)}\,x, \quad z \in G = \mathbb{C} \setminus \{0, 1\};$$

this solution has branch-points at $z = 1$ and the origin. Luckily, we shall have no need to study this *monodromy behavior* of solutions for a general multiply connected region. Instead, it will be sufficient to consider the simplest type of such regions, namely *punctured discs*. Assuming for simplicity that we have a punctured disc about the origin, the corresponding Riemann surface – or to be exact, the universal covering surface – is the Riemann surface of the (natural) logarithm. We require the reader to have some intuitive understanding of this concept, but we shall also discuss this surface on p. 226 in the Appendix.

Most of the time we shall restrict ourselves to systems of first-order linear equations. Since every νth order equation can be rewritten as a system (see Exercise 5 on p. 4), our results carry over to such equations as well. However, in some circumstances scalar equations are easier to handle than systems. So for practical purposes, such as computing power series solutions, we do not recommend to turn a given scalar equation into a system, but instead one should work with the scalar equation directly.

Many books on ordinary differential equations contain at least a chapter or two dealing with ODE in the complex plane. Aside from the books of Sibuya and Wasow, already mentioned in the introduction, we list the following more recent books in chronological order: *Ince* [138], *Bieberbach* [52], *Schäfke and Schmidt* [236], and *Hille* [120].

1.1 Simply Connected Regions

Throughout this chapter, we consider a system of the form

$$x' = A(z)\,x, \qquad z \in G, \tag{1.1}$$

where $A(z) = [a_{kj}(z)]$ denotes a $\nu \times \nu$ matrix whose entries are *holomorphic functions in some fixed region* $G \subset \mathbb{C}$, which we here assume to be *simply connected.* It is notationally convenient to think of such a matrix $A(z)$ as a *holomorphic matrix-valued function* in G.

Since we know from the theory of functions of a complex variable that such functions, if (once) differentiable in an open set, are automatically holomorphic there, it is obvious that solutions $x(z)$ of (1.1) are always vector-valued holomorphic functions. However, it is not clear off-hand that a solution always is holomorphic in *all of* the region G, but we shall prove this here. To begin, we show the following weaker result, which holds for arbitrary regions G.

Lemma 1 *Let a system* (1.1), *with* $A(z)$ *holomorphic in a region* $G \subset \mathbb{C}$, *be given. Then for every* $z_0 \in G$ *and every* $x_0 \in \mathbb{C}^\nu$, *there exists a unique vector-valued function* $x(z)$, *holomorphic in the largest disc* $D = D(z_0, \rho) = \{z\,:\,|z - z_0| < \rho\}$ *contained in* G, *such that*

$$x'(z) = A(z)\,x(z), \quad z \in D, \quad x(z_0) = x_0.$$

Hence we may say for short that every initial value problem has a unique solution that is holomorphic near z_0.

Proof: Assume for the moment that we were given a solution $x(z)$, holomorphic in D. Then the coordinates of the vector function $x(z)$ all can be expanded into power series about z_0, with a radius of convergence at least ρ. Combining these series into a vector, we can expand $x(z)$ into a *vector power series*[1]

$$x(z) = \sum_0^\infty x_n\,(z-z_0)^n, \quad |z-z_0| < \rho, \tag{1.2}$$

where $x_n \in \mathbb{C}^\nu$ are the *coefficient vectors.* Likewise, we can expand the coefficient matrix $A(z)$ into a *matrix power series*

$$A(z) = \sum_0^\infty A_n\,(z-z_0)^n, \quad |z-z_0| < \rho, \tag{1.3}$$

with *coefficient matrices* $A_n \in \mathbb{C}^{\nu\times\nu}$. Inserting these expansions into the system (1.1) and comparing coefficients leads to the identities

$$(n+1)\,x_{n+1} = \sum_{m=0}^{n} A_{n-m}\,x_m, \quad n \ge 0. \tag{1.4}$$

Hence, given x_0, we can recursively compute x_n for $n \ge 1$ from (1.4), which proves the uniqueness of the solution. To show existence, it remains to check whether the *formal power series solution* of our initial value problem, resulting from (1.4), converges for $|z-z_0| < \rho$. To do this, note that convergence of (1.3) implies $\|A_n\| \le c\,K^n$ for every constant $K > 1/\rho$ and sufficiently large $c > 0$, depending on K. Hence, equation (1.4) implies $(n+1)\,\|x_{n+1}\| \le c\sum_{m=0}^n K^{n-m}\,\|x_m\|$, $n \ge 0$. Defining a *majorizing sequence* (c_n) by $c_0 = \|x_0\|$, resp. $(n+1)\,c_{n+1} = c\sum_{m=0}^n K^{n-m}\,c_m$, $n \ge 0$, we conclude by induction that $\|x_n\| \le c_n$, for $n \ge 0$. The power series $f(z) = \sum_0^\infty c_n\,z^n$ can be easily checked to formally satisfy the linear ODE $y' = c\,(1-Kz)^{-1}\,y$. This equation has the solution $y(z) = c_0\,(1-Kz)^{-c/K}$, which is holomorphic in the disc $|z| < 1/K$. Expanding this function into its power series about the origin, inserting into the ODE and comparing coefficients, one checks that the coefficients satisfy the same recursion relation as the c_n, hence are, in fact, equal to the numbers c_n. This proves that the radius of convergence of (1.2) is at least $1/K$, and since K was arbitrary except for $K > 1/\rho$, the proof is completed.[2] □

Using the above *local result* together with the monodromy theorem on p. 225 in the Appendix, it is now easy to show the following *global version* of the same result:

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[1]Observe that whenever we write $|z-z_0| < \rho$, or a similar condition on z, we wish to state that the corresponding formula holds, and here in particular the series converges, for such z.

[2]An alternative proof for convergence of $f(z)$ is as follows: Show $(n+1)\,c_{n+1} = c\sum_{m=0}^n K^{n-m}\,c_m = (c+Kn)\,c_n$, hence the quotient test implies convergence of $f(z)$ for $|z| < 1/K$. While this argument is simpler, it depends on the structure of the recursion relation for (c_n) and fails in more general cases; see, e.g., the proof of Lemma 2 (p. 28).